Homework Solutions #5

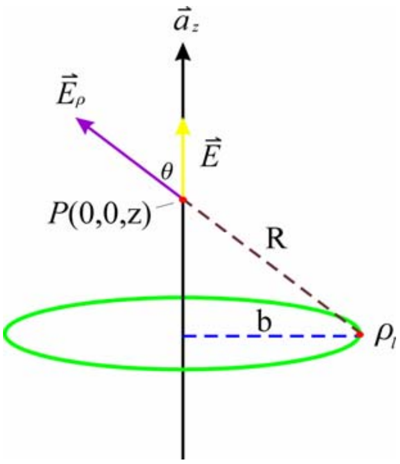
(Due date: 2011/4/11)

This problem set covers materials of Lesson 6. The full score is 80 points.

1) Consider a ring of radius b (m) placed on the *xy*-plane and centered at the origin. The ring has a uniform line charge density ρ_l (C/m).

1a) (10%) Determine the electric field intensity $\vec{E}$ (V/m) for an arbitrary point $P(0,0,z)$ on the *z*-axis.

Answer:



According to the figure shown above, we assume that the green ring of radius *b* has line density of ρ_l, and *z* is some observation point *P* on the *z*-axis. An infinitesimal arc of length $bd\phi$ with charge $\rho_l bd\phi$ generates a field $\vec{E}_\rho$ at *P*, where $\vec{E}_\rho$ makes an angle $\theta = \tan^{-1}(b/z)$ with the *z*-axis. From eq. (3-40) in the David K. Cheng's book, we can get $\left|\vec{E}_\rho\right| = E_\rho = \dfrac{\rho_l bd\phi}{4\pi\varepsilon_0 R^2}$. Because of the ϕ-symmetry, the total field $\vec{E}$ only has $\vec{a}_z$ component.

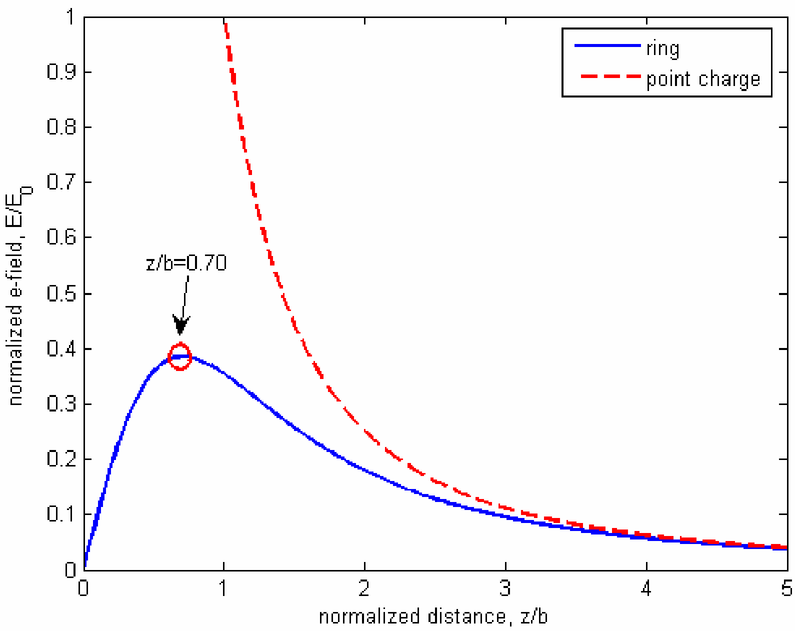
$$\vec{E} = \vec{a}_z \int_0^{2\pi} E_\rho \cos\theta = \vec{a}_z \int_0^{2\pi} \frac{\rho_l bd\phi}{4\pi\varepsilon_0 R^2}\frac{z}{R} = \vec{a}_z \frac{b\rho_l z}{2\varepsilon_0 R^3} = \boxed{\vec{a}_z \frac{\rho_l bz}{2\varepsilon_0\left(z^2+b^2\right)^{\frac{3}{2}}}}.$$

1b) (10%) According to the result of Problem 1a, plot the normalized magnitude of the electric field on the *z*-axis:

$$E'(z) \equiv \left|\vec{E}(0,0,z)\right| / E_0 \text{ , for } 0 \le z/b \le 5 \text{ ,}$$

where $E_0 = \dfrac{\rho_l}{2\varepsilon_0 b}$ (V/m). Mark the value of z/b where maximum electric field occurs. For comparison, also plot $E'(z)$ for $1 \le z/b \le 5$ due to a point charge $q = \rho_l \cdot 2\pi b$ at the origin in the same figure. Discuss the two results.

Answer:



The maximal E-field due to a ring occurs when $z/b = 0.7$.

According to the figure shown above, when the distance is far enough, the electric field due to the ring can be approximated by that due to a point charge. Actually the electric fields along the *z*-axis contributed by all small sections of the ring tend to cancel with one another when the observation point is close to the ring center. This causes deviation from the behavior of a point charge.

2) Consider the geometry of Fig. 3-40 of the text book (p147). Shift the finite line charge in the -x direction such that the line ranges $-L/2 < x < L/2$.

2a) (10%) Deduce the electric potential $V(x,b)$. (You are required to show the deduction process in detail.)

Answer:

From the eq. (3-63) of text book, we can know electric potential for a line charge,

$$V=\frac{1}{4\pi\varepsilon_0}\int_{L'}\frac{\rho}{R}dl'=\frac{1}{4\pi\varepsilon_0}\int_{-L/2}^{L/2}\frac{\rho_l}{R}dx'=\frac{\rho_l}{4\pi\varepsilon_0}\int_{-L/2}^{L/2}\frac{1}{\sqrt{(x'-x)^2+b^2}}dx'=\frac{\rho_l}{4\pi\varepsilon_0}\sinh^{-1}\left(\frac{x'-x}{b}\right)\Bigg|_{-L/2}^{L/2}$$

$$=\frac{\rho_l}{4\pi\varepsilon_0}\left[\sinh^{-1}\left(\frac{L/2-x}{b}\right)-\sinh^{-1}\left(\frac{-L/2-x}{b}\right)\right]=\frac{\rho_l}{4\pi\varepsilon_0}\left[\sinh^{-1}\left(\frac{L-x}{b}\right)+\sinh^{-1}\left(\frac{x}{b}\right)\right],$$

where the following integral formula is used:

$$\int\frac{1}{\sqrt{x^2+a^2}}dx=\int\frac{1}{\sqrt{a^2\sinh^2\theta+a^2}}d(a\sinh\theta)=\int\frac{1}{a\cosh\theta}\times a\cosh\theta d\theta$$

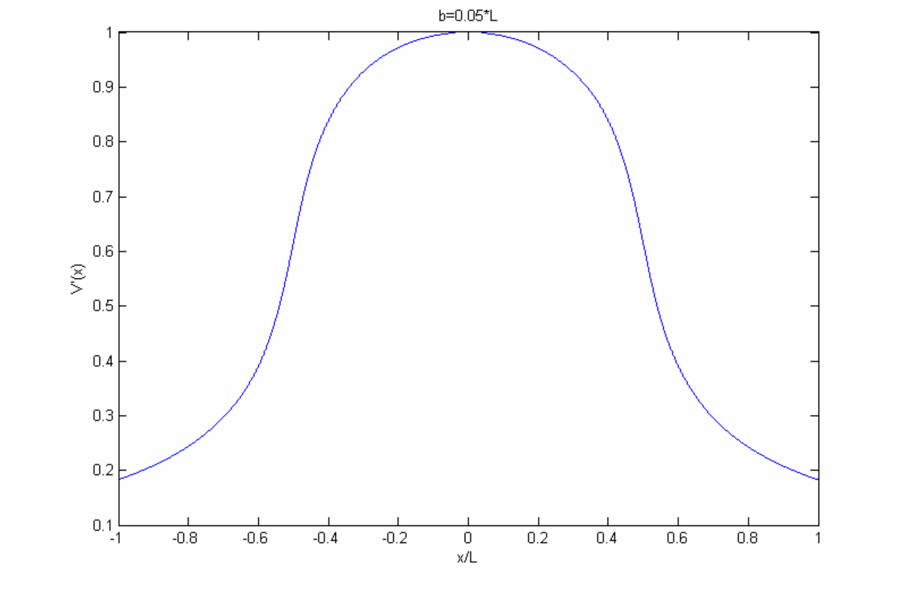
$$=\theta+c=\sinh^{-1}\frac{x}{a}+c.$$

2b) (5%) Plot the normalized electric potential $V'(x)\equiv V(x,b)/V(x=0,b)$ for $-1\le x/L\le 1$ and $b=0.05L$ (close to the line). (*Hint*: You can write a computer program to get the results by numerical integration even the analytic solution of $V(x,b)$ cannot be derived.)

Answer:

$$V'(-L\le x\le L)\equiv\frac{V(x,b)}{V(x=0,b)}=\frac{\sinh^{-1}(10-20x/L)+\sinh^{-1}(10+20x/L)}{2\times\sinh^{-1}(10)}$$

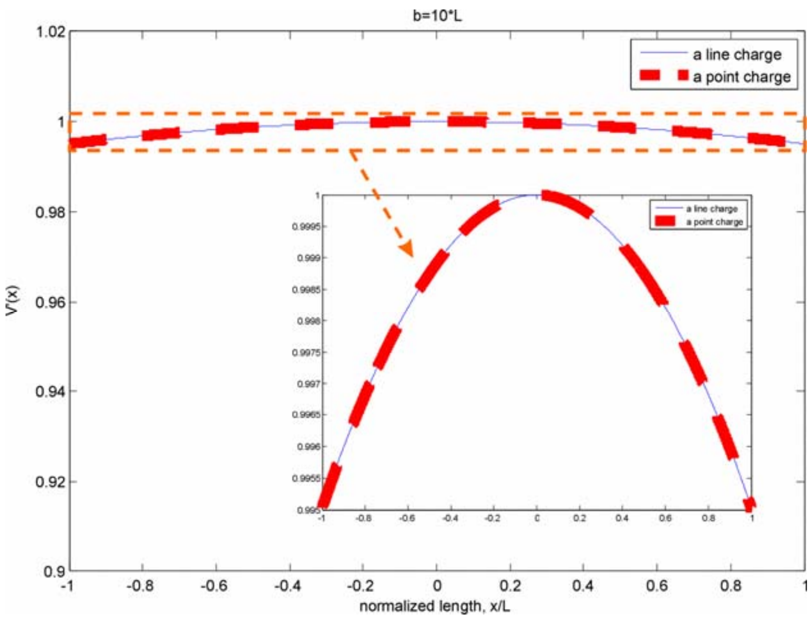
Edited by: Shang-Da Yang



2c) (5%) Plot the normalized electric potential $V'(x)$ for $-1 \le x/L \le 1$ and $b = 10L$ (far away from the line).

Answer:

$$V'(-L \le x \le L) \equiv \frac{V(x,b)}{V(x=0,b)} = \frac{\sinh^{-1}(0.05 - x/10L) + \sinh^{-1}(0.05 + x/10L)}{2 \times \sinh^{-1}(0.05)}$$



2d) (5%) Discuss the results of Problem 2b and 2c.

Answer:

The potential due to a finite line charge resembles that due to a point charge $\left(V(x) \approx \dfrac{\rho_l L}{4\pi\varepsilon_0\sqrt{x^2+b^2}}\right)$ when the observation point is far away from the line ($b >> L$).

It resembles that due to an infinite line charge $\left(V(x) \approx \text{constatnt}\right)$ if the observation point is very close to the line center ($b << L$ and $x \approx 0$).

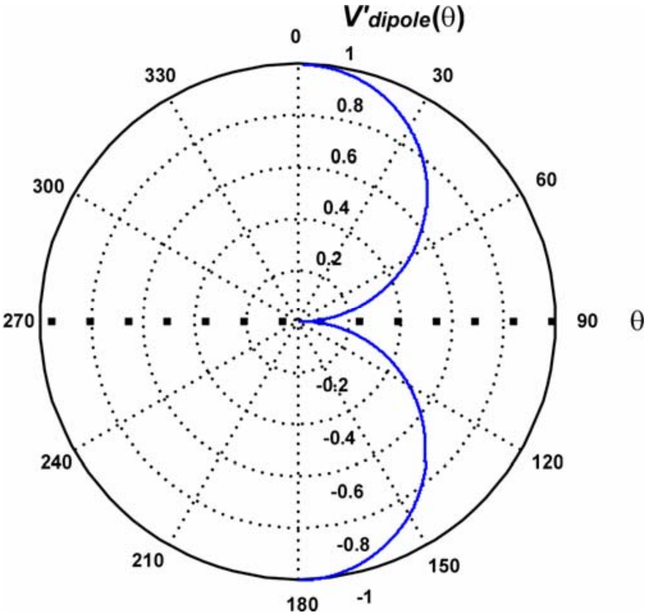
3) Consider an electric dipole shown in Fig. 6-5 of the lecture notes.

3a) (5%) According to eq. (6.18) of the lecture notes, plot the normalized electric potential:

$$V'_{dipole}(\theta) \equiv \frac{V(R_0,\theta,\phi_0)}{V(R_0,0,\phi_0)}, \text{ for } R_0 = 10d, \ \phi_0 = 0, \ 0 \le \theta \le \pi.$$

(*Hint*: Use the *Matlab* command ”polar” to plot the curve in the polar coordinate.)

Answer:



Note: $V'_{dipole}(\theta) < 0$, for $\frac{\pi}{2} \le \theta \le \pi$.

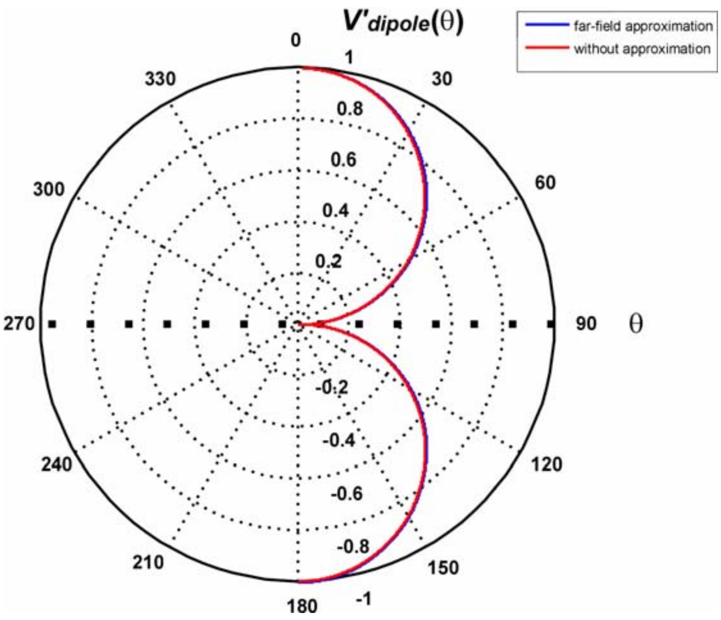
3b) (15%) To test the validity of eq. (6.18) of the lecture notes, write a program to evaluate and plot $V'_{dipole}(\theta)$ for $R_0 = 10d$, $\phi_0 = 0$, $0 \le \theta \le \pi$ according to the formula:

$$V(R) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right).$$

Show the two curves of Problem 3a and 3b in the same figure, and discuss the results.

Answer:

First use the cosine theorem $c^2 = a^2 + b^2 - 2ab\cos(\theta)$ to evaluate the R_+ and R_-.

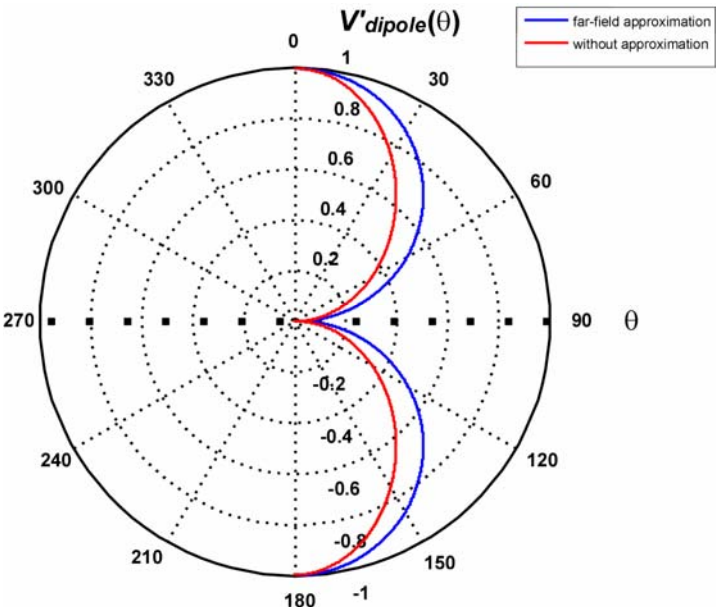


From the fugure we can see that $R_0 = 10d$ is far enough to apply the far-field approximation.

3c) (15%) With the program, you can investigate the potential in the vicinity of the dipole. Plot two curves of $V'_{dipole}(\theta)$ for $R_0 = 1.5d$, $\phi_0 = 0$, $0 \le \theta \le \pi$; obtained by eq. (6.18) of the lecture notes (like Problem 3a), and the program (like Problem 3b), respectively.

Show them in the same figure, and discuss the results.

Answer:



The two curves do not match with each other, and the far-field approximation fails at $R_0 = 1.5d$.