Chapter 12
Introduction To The Laplace Transform

12.1 Definition of the Laplace Transform
12.2-3 The Step & Impulse Functions
12.4 Laplace Transform of specific functions
12.5 Operational Transforms
12.6 Applying the Laplace Transform
12.7 Inverse Transforms of Rational Functions
12.8 Poles and Zeros of $F(s)$
12.9 Initial- and Final-Value Theorems

Overview

- Laplace transform is a technique that is particularly useful in linear circuit analysis when:

1. Considering transient response (e.g. switching) of circuits with multiple nodes and meshes.
2. The sources are more complicated than the simple dc level jumps.
3. Introducing the concept of transfer function to analyze frequency-dependent sinusoidal steady-state response (Chapters 13, 14).

Key points

- What is the definition of the Laplace transform?
- What are the Laplace transforms of unit step, impulse, exponential, and sinusoidal functions?
- What are the Laplace transforms of the derivative, integral, shift, and scaling of a function?
- How to perform partial fraction expansion for a rational function $F(s)$ and perform the inverse Laplace transform?

Section 12.1
Definition of the Laplace Transform

What is Laplace transform?

- Transforming a real function $f(t)$ of real variable t to a complex function $F(s)$ of complex variable s:

$$F(s) = L\{f(t)\} \equiv \int_{0^-}^{\infty} f(t)\boxed{e^{-st}}dt \in C.$$

kernel

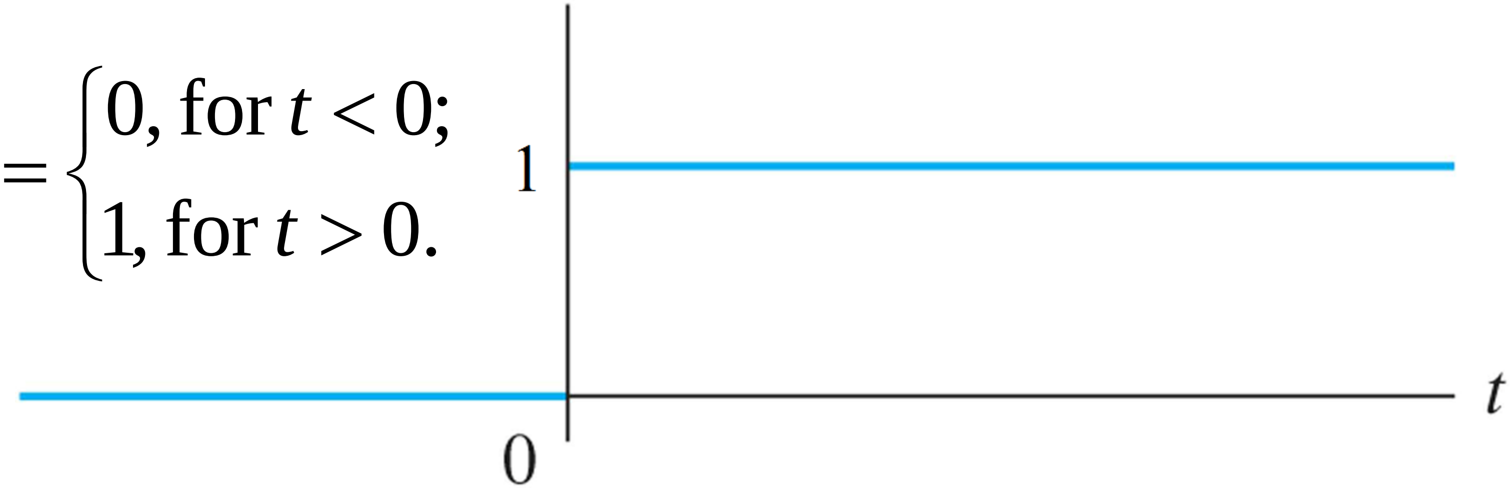
- The integral will converge (1) over a portion of the s-plane (e.g. $\mathrm{Re}(s) > 0$), and (2) for most of the functions except for those of little interest (e.g. t^t).
- $F(s)$ is determined by $f(t)$ only for $t > 0^-$. Thus we use it to predict the response after initial conditions have been established.

Section 12.2, 12.3
The Step and Impulse Functions

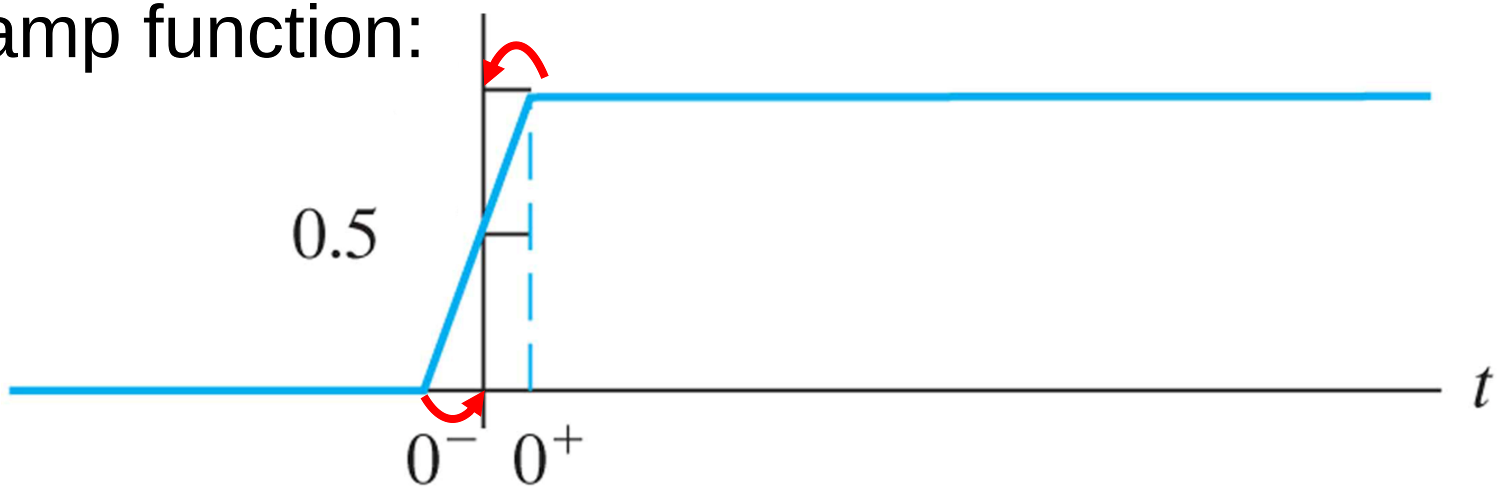
1. Definition of unit step function $u(t)$
2. Definition of impulse function $\delta(t)$
3. Laplace transforms of $\delta(t)$ and $\delta'(t)$

The unit step function $u(t)$

$$u(t)=\begin{cases}0, \text{ for } t<0;\\ 1, \text{ for } t>0.\end{cases}$$

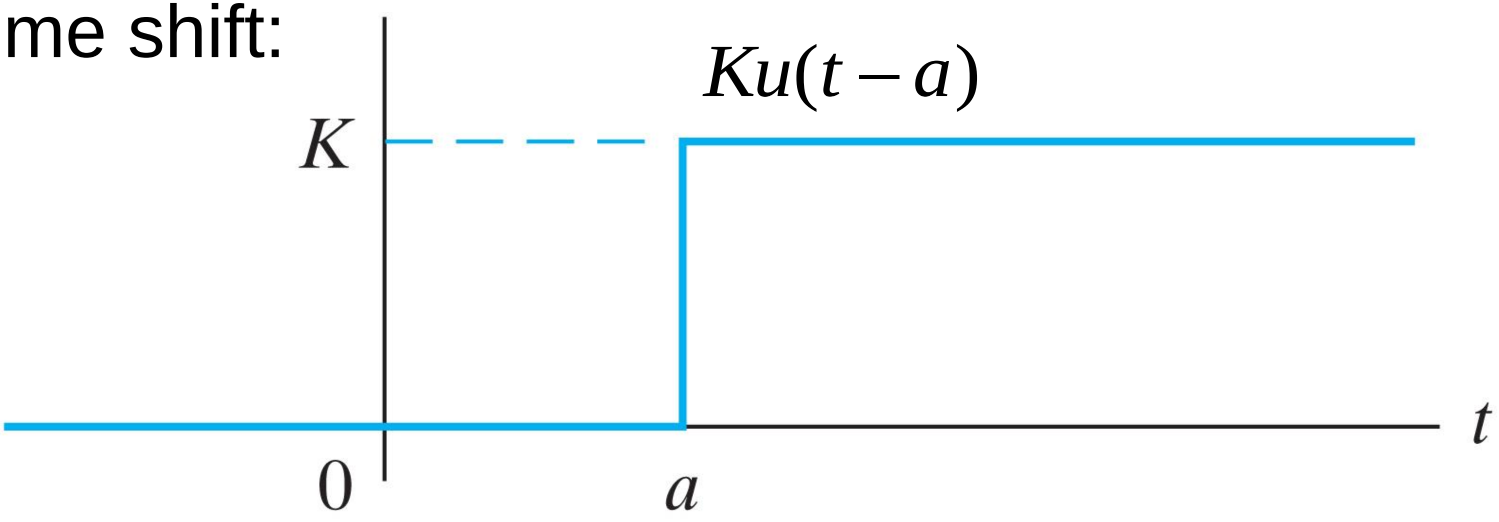


- $u(t)$ can be approximated by the limit of a linear ramp function:

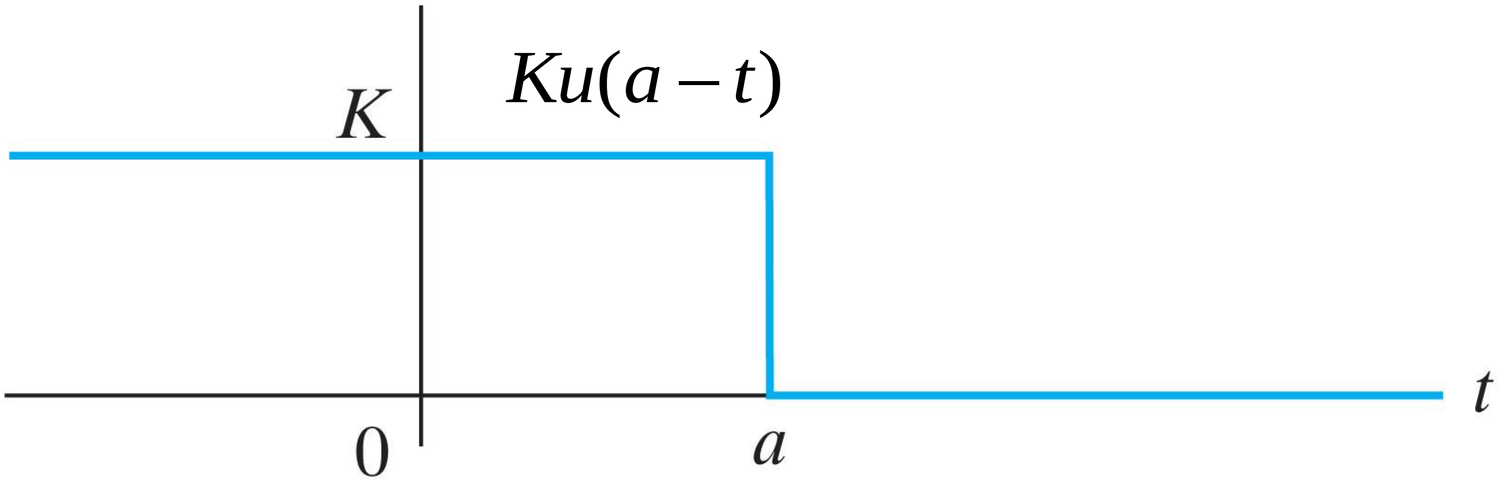


Representation of time shift and reversal

- Time shift:

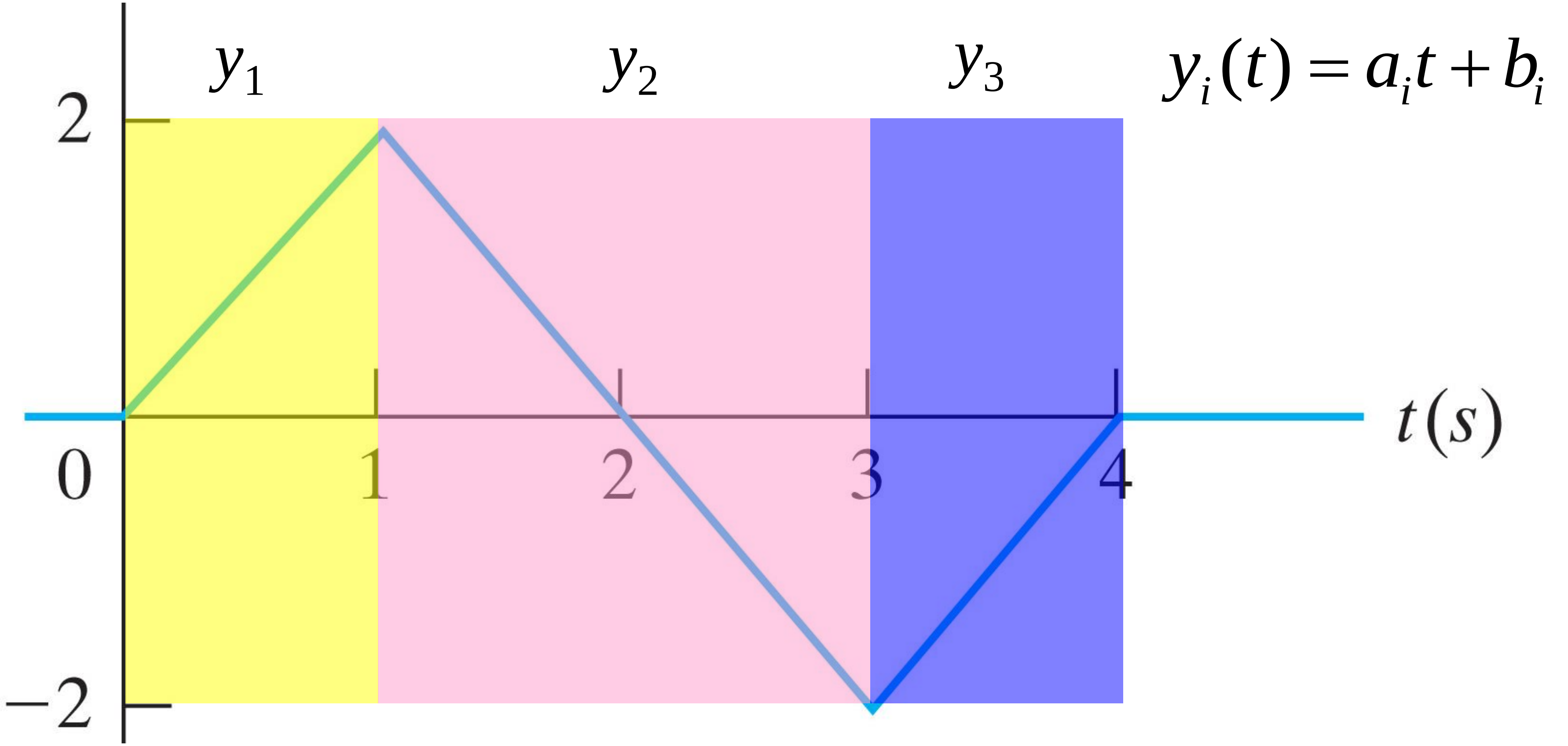


- Time reversal:



Example 12.1: A pulse of finite width (1)

- Q: Express the piecewise linear function $f(t)$ as superposition of 3 functions.



Example 12.1 (2)

- For each interval, $f(t)$ can be expressed as the product of a linear function and a square pulse (difference between two step functions).
- For example, for $1 < t < 3$, the corresponding linear and square pulse functions are:

$$y_2(t) = -2t + 4, \text{ and } p_2(t) = u(t-1) - u(t-3).$$

- The entire function can be represented by:

$$f(t) = y_1(t)p_1(t) + y_2(t)p_2(t) + y_3(t)p_3(t).$$

The impulse function δ(*t*)

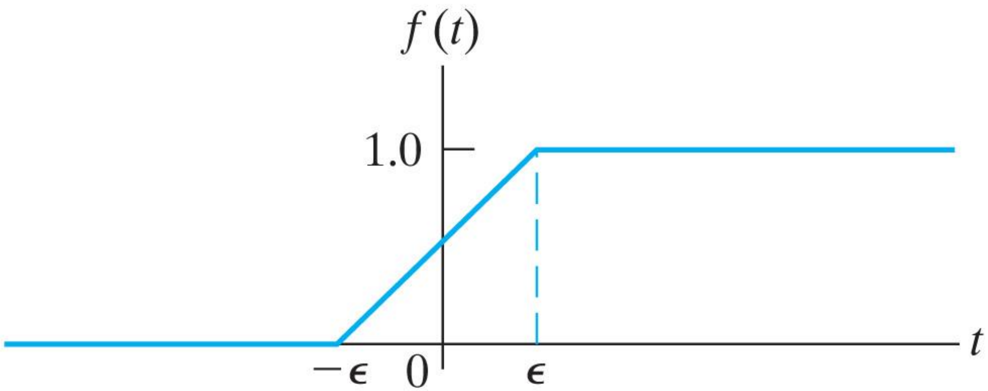
- An idealized math representation of sharply peaked stimulus:

$$\delta(t) = \begin{cases} \infty, & t = 0; \\ 0, & \text{otherwise}; \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

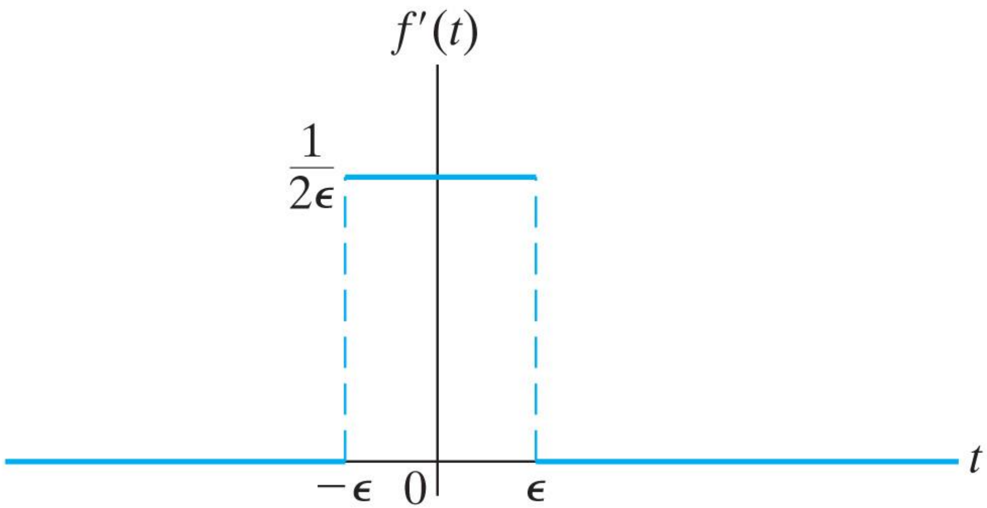
- δ(*t*) has (1) zero duration, (2) infinite peak amplitude, (3) unit area (strength).

$\delta(t)$ is the derivative of $u(t)$



(a)

$$u(t) = \lim_{\varepsilon \to 0} f(t)$$



(b)

$$\delta(t) = \lim_{\varepsilon \to 0} f'(t) = u'(t)$$

The sifting property of δ-function

- Sampling of $f(t)$ at $t=a$ (>0) can be formulated by integral of $f(t)$ times $\delta(t\text{-}a)$:

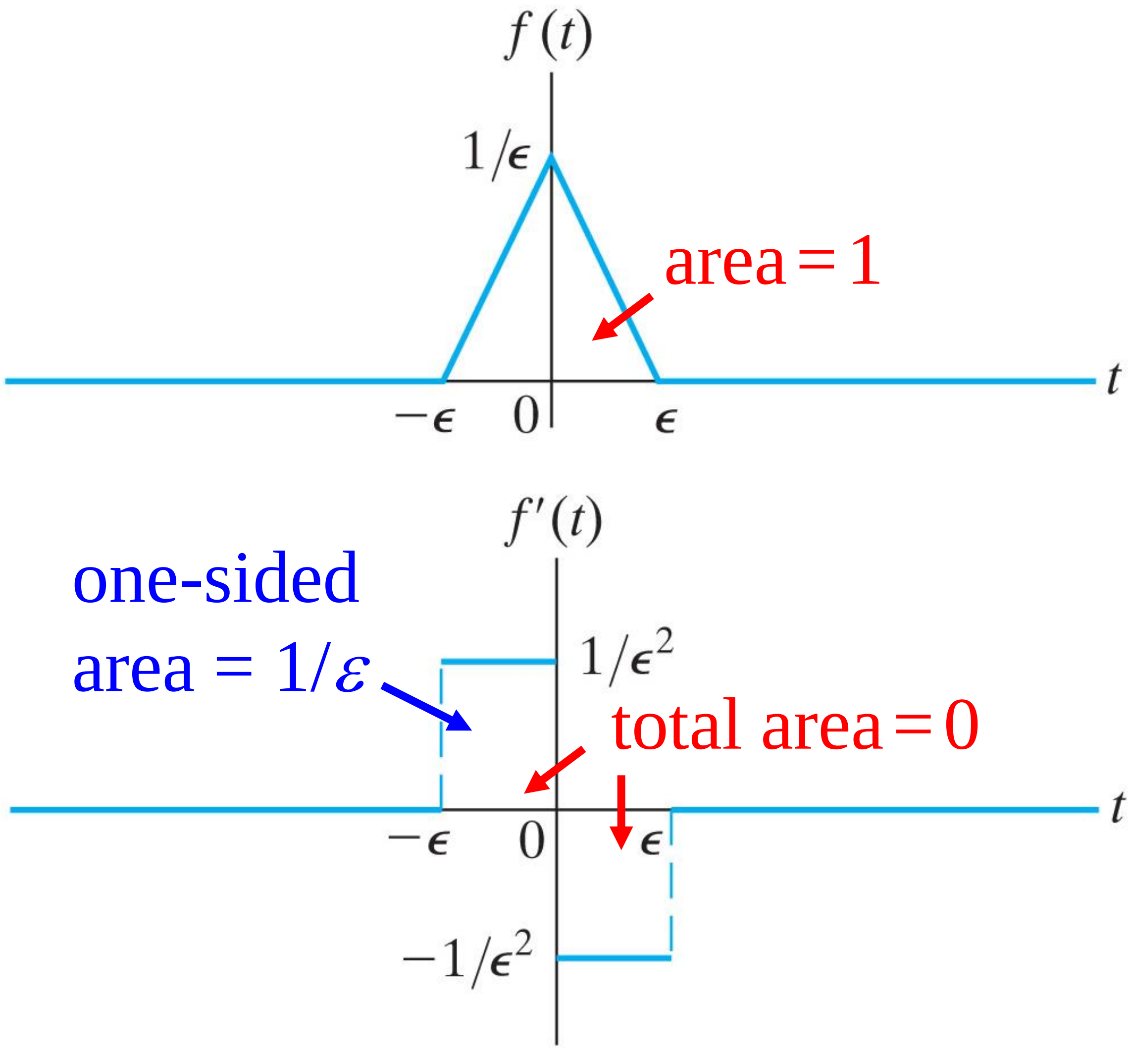
$$\int_{0^-}^{\infty} f(t)\delta(t-a)dt = \lim_{\varepsilon\to 0}\int_{a-\varepsilon}^{a+\varepsilon} f(t)\delta(t-a)dt$$

$$= f(a)\left[\lim_{\varepsilon\to 0}\int_{a-\varepsilon}^{a+\varepsilon}\delta(t-a)dt\right] = f(a).$$

- It can be used in calculating the Laplace transform of a δ-function:

$$L\{\delta(t)\} = \int_{0^-}^{\infty}\delta(t)e^{-st}dt = e^{-s(0)} = 1, \text{ for any } s\in C.$$

The derivative of δ-function

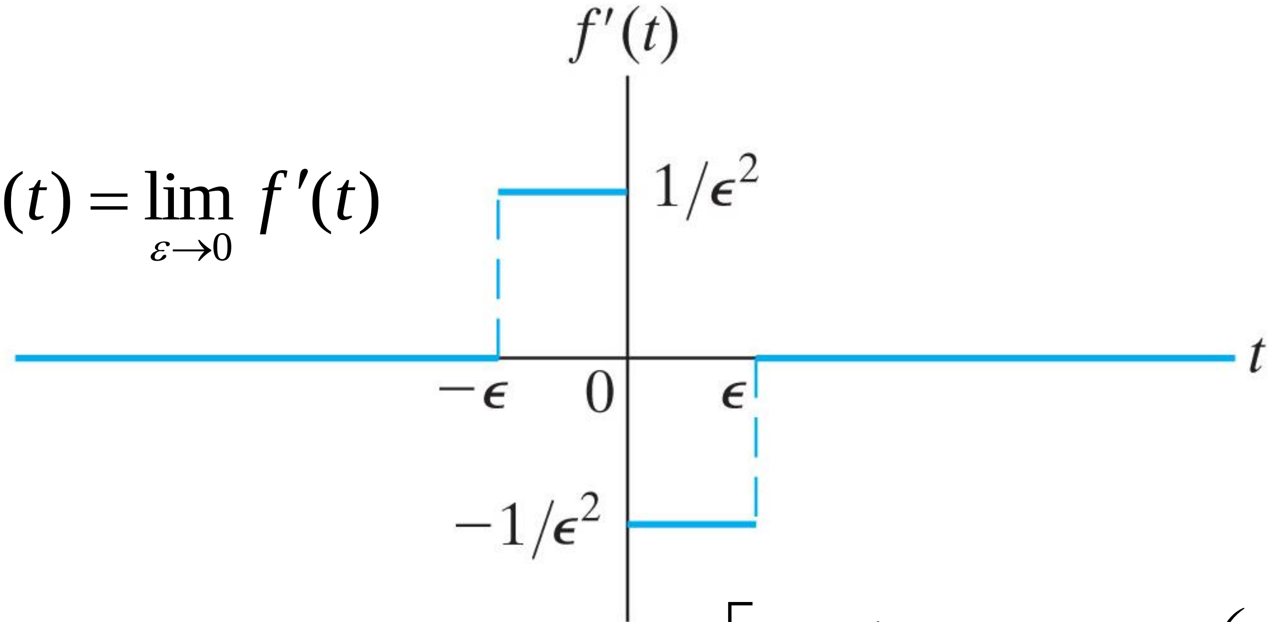


$$\delta(t) = \lim_{\varepsilon \to 0} f(t)$$

$$\delta'(t) = \lim_{\varepsilon \to 0} f'(t)$$

Laplace transform of δ′(t)

$$\delta'(t) = \lim_{\varepsilon \to 0} f'(t)$$



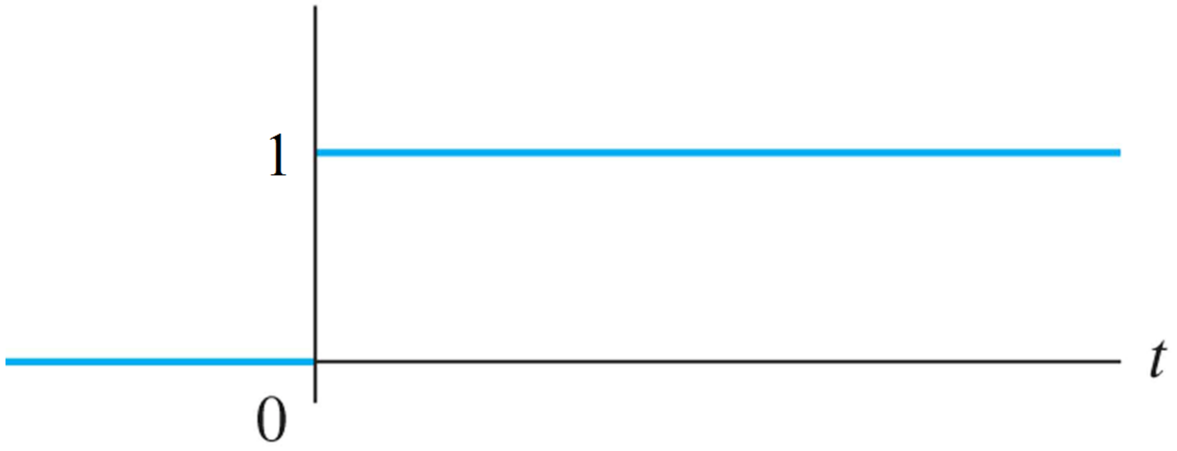
$$L\{\delta'(t)\} = \lim_{\varepsilon \to 0} \int_{0^-}^{\infty} f'(t) e^{-st} dt = \lim_{\varepsilon \to 0} \left[\int_{-\varepsilon}^{0^-} \frac{1}{\varepsilon^2} e^{-st} dt + \int_{0^+}^{\varepsilon} \left(\frac{-1}{\varepsilon^2} \right) e^{-st} dt \right]$$

$$= \lim_{\varepsilon \to 0} \frac{e^{s\varepsilon} + e^{-s\varepsilon} - 2}{s\varepsilon^2} = \lim_{\varepsilon \to 0} \frac{(e^{s\varepsilon} - e^{-s\varepsilon})s}{2s\varepsilon} = \lim_{\varepsilon \to 0} \frac{(e^{s\varepsilon} + e^{-s\varepsilon})s^2}{2s} = s.$$

- Even δ′(t) has well defined Laplace transform!

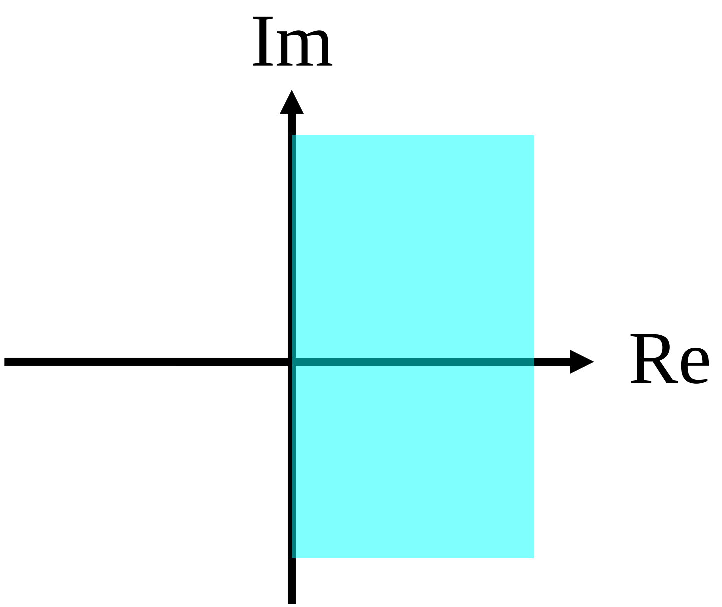
Section 12.4
Laplace Transform of Specific Functions

E.g. Unit step function

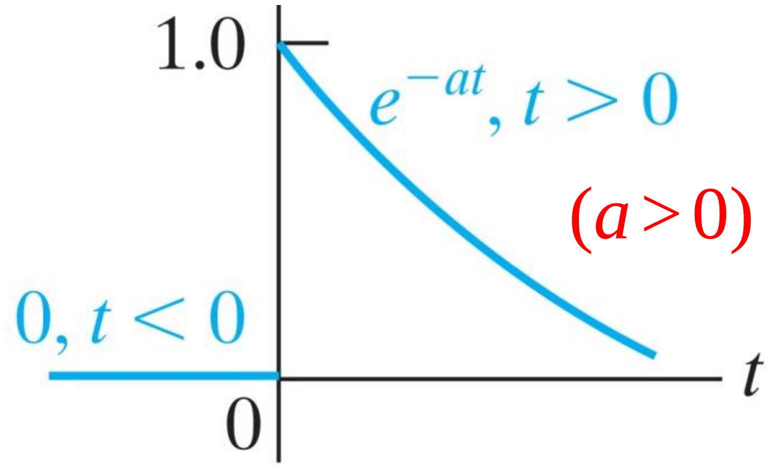


$$L\{u(t)\} = \int_{0^-}^{\infty} u(t)e^{-st}dt = \int_{0^+}^{\infty} 1e^{-st}dt$$

$$= \left.\frac{e^{-st}}{-s}\right|_{0^+}^{\infty} = \boxed{\frac{1}{s}}, \text{ if } \underline{\mathrm{Re}\{s\} > 0.}$$

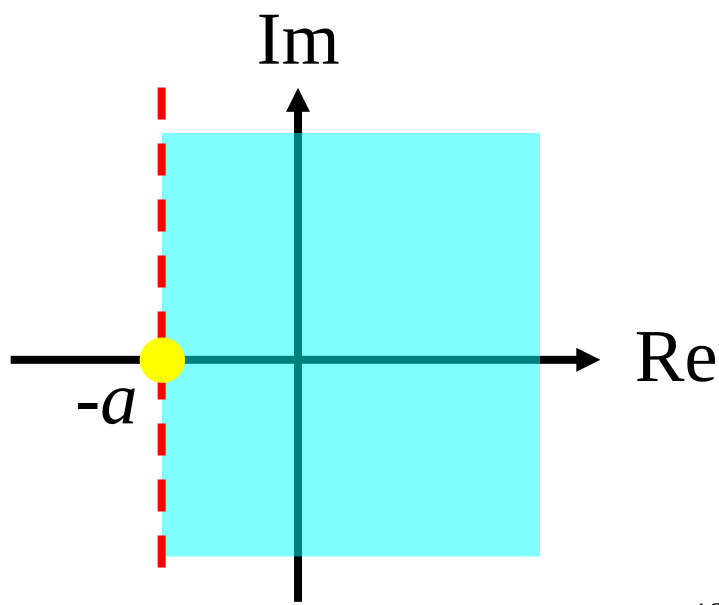


E.g. Single-sided exponential function

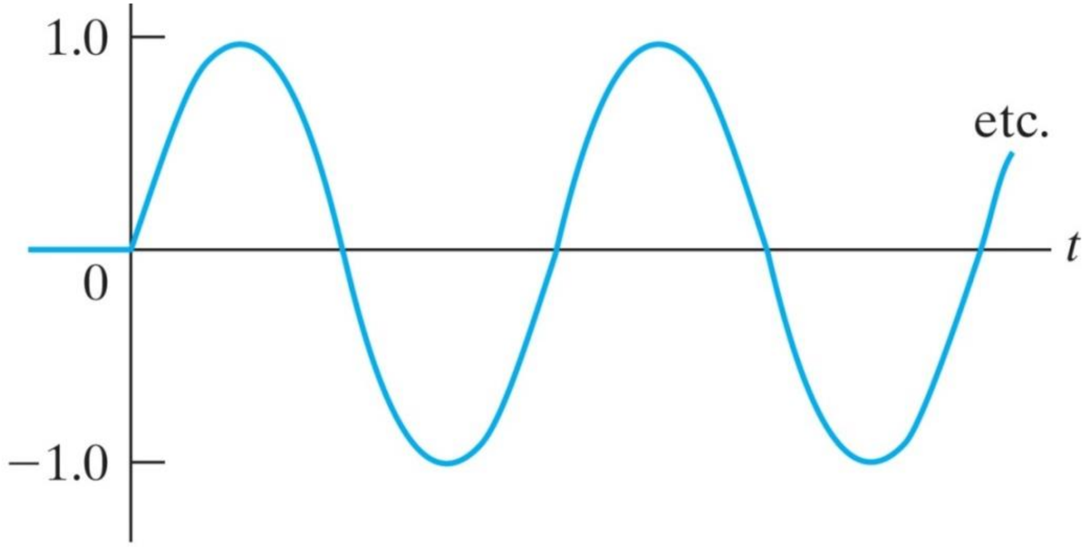


$$L\{e^{-at}\} = \int_{0^-}^{\infty} e^{-at} e^{-st} dt = \int_{0^-}^{\infty} e^{-(a+s)t} dt$$

$$= \left. \frac{e^{-(a+s)t}}{-(a+s)} \right|_{0^-}^{\infty} = \boxed{\frac{1}{s+a}}, \text{ if } \underline{\mathrm{Re}\{s\} > -a.}$$



E.g. Sinusoidal function



$$L\{\sin \omega t\} = \int_{0^-}^{\infty} (\sin \omega t) e^{-st} dt = \int_{0^-}^{\infty} \frac{e^{j\omega t} - e^{-j\omega t}}{2j} e^{-st} dt$$

$$\frac{1}{2j} \int_{0^-}^{\infty} \left[e^{-(s-j\omega)t} - e^{-(s+j\omega)t} \right] dt = \frac{1}{2j} \left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right) = \boxed{\frac{\omega}{s^2 + \omega^2}}.$$

List of Laplace transform pairs (1)

Type	$f(t)$	$F(s)$
impulse	$\delta(t)$	1
step	$u(t)$	$\dfrac{1}{s}$
ramp	$tu(t)$	$\dfrac{1}{s^2}$
exponential	$e^{-at}u(t)$	$\dfrac{1}{s+a}$

- Laplace transform of polynomial functions:

$$L\{t^2\} = \frac{2}{s^3},$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}.$$

List of Laplace transform pairs (2)

Type	$f(t)$	$F(s)$
sine	$\sin \omega t \cdot u(t)$	$\dfrac{\omega}{s^2+\omega^2}$
cosine	$\cos \omega t \cdot u(t)$	$\dfrac{s}{s^2+\omega^2}$
damped ramp	$te^{-at} \cdot u(t)$	$\dfrac{1}{(s+a)^2}$
damped sine	$e^{-at} \sin \omega t \cdot u(t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$

- Damping by exponential decay function causes a shift along the real axis in the s-domain.

Section 12.5
Operational Transforms

What are operational transforms?

- Operational transforms indicate how the mathematical operations performed on either $f(t)$ or $F(s)$ are converted into the opposite domain.
- Useful in calculating the Laplace transform of a function $g(t)$ derived by performing some math operation on $f(t)$ with known $F(s)$.

First-order time derivative

$$L\{f'(t)\} = \int_{0^-}^{\infty} f'(t) e^{-st} dt$$

integration by parts

$$= \left[f(t) e^{-st} \Big|_{0^-}^{\infty} \right] - \int_{0^-}^{\infty} f(t)(-se^{-st}) dt$$

$$= \left[\boxed{0} - f(0^-)\right] + s\int_{0^-}^{\infty} f(t) e^{-st} dt = sF(s) - f(0^-).$$

if $f(\infty)e^{-s(\infty)} = 0$

initial condition

- E.g. $\delta(t) = u'(t),\ L\{u(t)\} = \dfrac{1}{s},$

$$\Rightarrow L\{\delta(t)\} = s\frac{1}{s} - \cancel{u(0^-)} = 1.$$

Higher-order time derivatives

- 2nd-order derivative:

$$\text{Let } g(t) = f'(t), \ \Rightarrow G(s) = sF(s) - f(0^-).$$

$$L\{f''(t)\} = L\{g'(t)\} = sG(s) - g(0^-)$$

$$= s[sF(s) - f(0^-)] - f'(0^-) = s^2F(s) - sf(0^-) - f'(0^-).$$

initial conditions

- nth-order derivative:

$$L\{f^{(n)}(t)\} = s^nF(s) - [s^{n-1}f(0^-) + s^{n-2}f'(0^-) + \cdots + f^{(n-1)}(0^-)].$$

initial conditions

Time integral

$$L\left\{\int_{0^-}^{t} f(x)dx\right\} = \int_{0^-}^{\infty}\left[\int_{0^-}^{t} f(x)dx\right]e^{-st}dt = \int_{0^-}^{\infty} u(t)v'(t)dt, \text{ where}$$

$$u(t) = \int_{0^-}^{t} f(x)dx, \Rightarrow u'(t) = f(t);\ v'(t) = e^{-st}, \Rightarrow v(t) = \frac{e^{-st}}{-s}.$$

$$L\left\{\int_{0^-}^{t} f(x)dx\right\} = u(t)v(t)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} u'(t)v(t)dt$$

$$= \left[\int_{0^-}^{t} f(x)dx\right]\left(\frac{e^{-st}}{-s}\right)\Bigg|_{0^-}^{\infty} \boxed{-\int_{0^-}^{\infty} f(t)\left(\frac{e^{-st}}{-s}\right)dt}$$

$$= \left[\overset{\text{finite}}{\int_{0^-}^{\infty} f(x)dx}\right]\left(\frac{\overset{0}{e^{-s(\infty)}}}{-s}\right) - \cancel{\left[\int_{0^-}^{0^-} f(x)dx\right]}\left(\frac{\overset{1}{e^{-s(0^-)}}}{-s}\right) \boxed{+\frac{1}{s}F(s)} = \frac{F(s)}{s}.$$

- The formula is valid only if the function is integrable.

Scaling

$$L\{f(at)\} = \int_{0^-}^{\infty} f(at)e^{-st}dt; \text{ let } at = t',$$

$$\Rightarrow L\{f(at)\} = \int_{0^-}^{\infty} f(t')e^{-st'/a}\frac{dt'}{a} = \frac{F(s/a)}{a}, \text{ if } a > 0.$$

- Intuitively, a larger value of a corresponds to a narrower function in the time domain but a broader function in the frequency domain. The width of $f(at)$ times the width of $F(s/a)$ is a constant independent of a.

Translation in the t and s domains

- Translation in the time domain:

$$L\left\{f(t-a)u(t-a)\right\} = e^{-as}F(s),\ \text{for}\ a > 0.$$

- Translation in the frequency domain:

$$L\left\{f(t)e^{-at}\right\} = F(s+a).$$

- Both relations can be proven by change of variable of integration.

Section 12.7
Inverse Transforms of Rational Functions

1. Distinct real roots
2. Distinct complex roots
3. Repeated real roots
4. Repeated complex roots
5. Improper rational functions

Why only rational functions?

- For linear, lumped-parameter circuits with constant component parameters, the s-domain expression for $v(t)$, $i(t)$ are always rational functions, i.e. ratio of two polynomials:

$$F(s)=\frac{N(s)}{D(s)}=\frac{a_n s^n + a_{n-1}s^{n-1}+..+a_0}{b_m s^m + b_{m-1}s^{m-1}+..+b_0}.$$

- General inverse Laplace transform:

$$f(t)=L^{-1}\{F(s)\}=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}F(s)e^{st}ds,$$

involves with complex integral.

How to calculate?

- If $F(s)$ is a proper $(m > n)$ rational function, the inverse transform is calculated by (1) partial fraction expansion, (2) individual inverse transforms (4 types).
- If $F(s)$ is an improper $(m \leq n)$ rational function, decompose $F(s)$ as the summation of a polynomial function and a proper rational function, which are inverse transformed individually.

Type I: $D(s)$ has distinct real roots (1)

$$F(s)=\frac{96(s+5)(s+12)}{s(s+8)(s+6)}=\frac{K_1}{s}+\frac{K_2}{s+8}+\frac{K_3}{s+6}.$$

$$\underline{F(s)s\big|_{s=0}}=\left(K_1+K_2\frac{\cancel{s}}{s+8}+K_3\frac{\cancel{s}}{s+6}\right)\Bigg|_{s=0};$$

$$\Rightarrow\frac{96(s+5)(s+12)}{(s+8)(s+6)}\Bigg|_{s=0}=\frac{96(5)(12)}{(8)(6)}=120=\underline{K_1}.$$

$$\Rightarrow\begin{cases}K_2=F(s)(s+8)\big|_{s=-8}=-72,\\ K_3=F(s)(s+6)\big|_{s=-6}=48.\end{cases}$$

Type I: $D(s)$ has distinct real roots (2)

$$F(s)=\frac{120}{s}-\frac{72}{s+8}+\frac{48}{s+6},$$

$$f(t)=120L^{-1}\left\{\frac{1}{s}\right\}-72L^{-1}\left\{\frac{1}{s+8}\right\}+48L^{-1}\left\{\frac{1}{s+6}\right\}$$

$$=120L^{-1}\left\{\frac{1}{s}\right\}-72L^{-1}\left\{\frac{1}{s}\right\}e^{-8t}+48L^{-1}\left\{\frac{1}{s}\right\}e^{-6t}$$

$$=\left[120-72e^{-8t}+48e^{-6t}\right]u(t).$$

- The circuit is over-damped.

Type II: $D(s)$ has distinct complex roots (1)

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}, \Rightarrow \text{roots of } D(s): \ s = -6, -3 \pm j4.$$

$$F(s) = \frac{K_1}{s+6} + \frac{K_2}{s+3-j4} + \frac{K_3}{s+3+j4},$$

$$K_1 = F(s)(s+6)\Big|_{s=-6} = -12,$$

$$K_2 = F(s)(s+3-j4)\Big|_{\boxed{s=-3+j4}} = \frac{100(s+3)}{(s+6)(s+3+j4)}\Bigg|_{s=-3+j4} = 6-j8,$$

$$K_3 = F(s)(s+3+j4)\Big|_{\boxed{s=-3-j4}} = 6+j8 = K_2^*.$$

- Conjugate roots must have conjugate coefficients.

Type II: $D(s)$ has distinct complex roots (2)

$$F(s)=\frac{-12}{s+6}+\frac{6-j8}{s+3-j4}+\frac{6+j8}{s+3+j4},$$

$$f(t)=L^{-1}\left\{\frac{-12}{s+6}\right\}+L^{-1}\left\{\frac{6-j8}{s+3-j4}\right\}+L^{-1}\left\{\frac{6+j8}{s+3+j4}\right\}$$

$$=\left[-12e^{-6t}+(6-j8)e^{-(3-j4)t}+(6+j8)e^{-(3+j4)t}\right]u(t)$$

$$=\left\{-12e^{-6t}+e^{-3t}\left[(6-j8)e^{j4t}+(6+j8)e^{-j4t}\right]\right\}u(t)$$

$$=\left\{-12e^{-6t}+e^{-3t}\cdot 2\,\mathrm{Re}\left[\underbrace{(6-j8)}_{10\angle-53^\circ}e^{j4t}\right]\right\}u(t)$$

$$=\left[-12e^{-6t}+20e^{-3t}\cos(4t-53^\circ)\right]u(t).$$

- The circuit shows over-damped (or 1st-order) and under-damped characteristics.

Type III: $D(s)$ has repeated real roots (1)

$$F(s)=\frac{100(s+25)}{s(s+5)^3}=\frac{K_1}{s}+\left[\frac{K_2}{(s+5)^3}+\frac{K_3}{(s+5)^2}+\frac{K_4}{s+5}\right].$$

highest order

$$\boxed{G(s)\equiv F(s)(s+5)^3}=\frac{100(s+25)}{s}$$

$$=K_1\frac{(s+5)^3}{s}+K_2+K_3(s+5)+K_4(s+5)^2;$$

$$\boxed{G(-5)}=\frac{100(20)}{(-5)}=-400=K_2.$$

Type III: $D(s)$ has repeated real roots (2)

$$F(s)=\frac{100(s+25)}{s(s+5)^3}=\frac{K_1}{s}+\left[\frac{K_2}{(s+5)^3}+\frac{K_3}{(s+5)^2}+\frac{K_4}{s+5}\right].$$

$$G(s)=\frac{100(s+25)}{s}=K_1\frac{(s+5)^3}{s}+K_2+K_3(s+5)+K_4(s+5)^2;$$

$$G'(s)=100\frac{s-(s+25)}{s^2}=\frac{-2500}{s^2}$$

$$=K_1\frac{3(s+5)^2 s-(s+5)^3}{s^2}+K_3+2K_4(s+5);$$

$$\boxed{G'(-5)}=\frac{-2500}{25}=100=K_3.$$

Type III: $D(s)$ has repeated real roots (3)

$$F(s)=\frac{100(s+25)}{s(s+5)^3}=\frac{K_1}{s}+\left[\frac{K_2}{(s+5)^3}+\frac{K_3}{(s+5)^2}+\frac{K_4}{s+5}\right].$$

$$G'(s)=\frac{-2500}{s^2}=K_1\frac{(s+5)^2(2s-5)}{s^2}+K_3+2K_4(s+5);$$

$$G''(s)=\frac{5000}{s^3}=K_1\frac{2(s+5)(s^2+5s-25)}{s^3}+2K_4;$$

$$\boxed{G''(-5)}=\frac{5000}{-125}=-40=\underline{2K_4},\ \ K_4=-20.$$

Type III: $D(s)$ has repeated real roots (4)

$$F(s) = \frac{20}{s} + \left[\frac{-400}{(s+5)^3} + \frac{100}{(s+5)^2} + \frac{-20}{s+5} \right].$$

$$f(t) = L^{-1}\left\{ \frac{20}{s} \right\} - L^{-1}\left\{ \frac{400}{(s+5)^3} \right\} + L^{-1}\left\{ \frac{100}{(s+5)^2} \right\} - L^{-1}\left\{ \frac{20}{s+5} \right\}$$

$$= \left[20 - 400\frac{t^2}{2!}e^{-5t} - 100\frac{t}{1!}e^{-5t} - 20e^{-5t} \right] u(t).$$

- The circuit shows critically-damped behavior.

Type IV: $D(s)$ has repeated complex roots (1)

$$F(s)=\frac{768}{(s^2+6s+25)^2}=\frac{768}{(s+3-j4)^2(s+3+j4)^2}$$

$$=\frac{K_1}{(s+3-j4)^2}+\frac{K_2}{s+3-j4}+\frac{K_1^*}{(s+3+j4)^2}+\frac{K_2^*}{s+3+j4}.$$

highest order

$$\boxed{G(s)\equiv F(s)(s+3-j4)^2}=\frac{768}{(s+3+j4)^2}$$

$$=K_1+K_2(s+3-j4)+K_1^*\frac{(s+3-j4)^2}{(s+3+j4)^2}+K_2^*\frac{(s+3-j4)^2}{s+3+j4};$$

$$\boxed{G(-3+j4)}=\frac{768}{(j8)^2}=-12=K_1.\ \Rightarrow K_2=\boxed{G'(-3+j4)}=-j3.$$

Type IV: $D(s)$ has repeated complex roots (2)

$$F(s) = \frac{-12}{(s+3-j4)^2} + \frac{-j3}{s+3-j4} + \frac{-12}{(s+3+j4)^2} + \frac{j3}{s+3+j4}$$

$$f(t) = L^{-1}\left\{\frac{-12}{(s+3-j4)^2}\right\} + c.c. + L^{-1}\left\{\frac{3\angle -90^\circ}{s+3-j4}\right\} + c.c.$$

$$= \left[-12te^{-3t}e^{j4t} + c.c + 3e^{-3t}e^{j(4t-90^\circ)} + c.c.\right]u(t)$$

$$= \left[-24te^{-3t}\cos 4t + 6e^{-3t}\cos(4t-90^\circ)\right]u(t)$$

$$= \left[-24te^{-3t}\cos 4t + 6e^{-3t}\sin 4t\right]u(t).$$

- The circuit shows critically- and under-damped characteristics.

Useful transformation pairs

$F(s)$	$f(t)$
$\dfrac{1}{s+a}$	$e^{-at}u(t)$
$\dfrac{1}{(s+a)^2}$	$te^{-at}u(t)$
$\dfrac{K}{s+\alpha-j\beta}+\dfrac{K^*}{s+\alpha+j\beta}$	$2\lvert K\rvert e^{-\alpha t}\cos(\beta t+\theta_K)u(t)$
$\dfrac{K}{(s+\alpha-j\beta)^2}+\dfrac{K^*}{(s+\alpha+j\beta)^2}$	$2\lvert K\rvert te^{-\alpha t}\cos(\beta t+\theta_K)u(t)$

Inverse transform of improper rational functions

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 4s + 20}$$

$$= \left(s^2 + 4s + 10\right) - \frac{20}{s+4} + \frac{50}{s+5}.$$

$$f(t) = \delta''(t) + 4\delta'(t) + 10\delta(t)$$

$$+ \left[-20e^{-4t} + 50e^{-5t}\right] u(t).$$

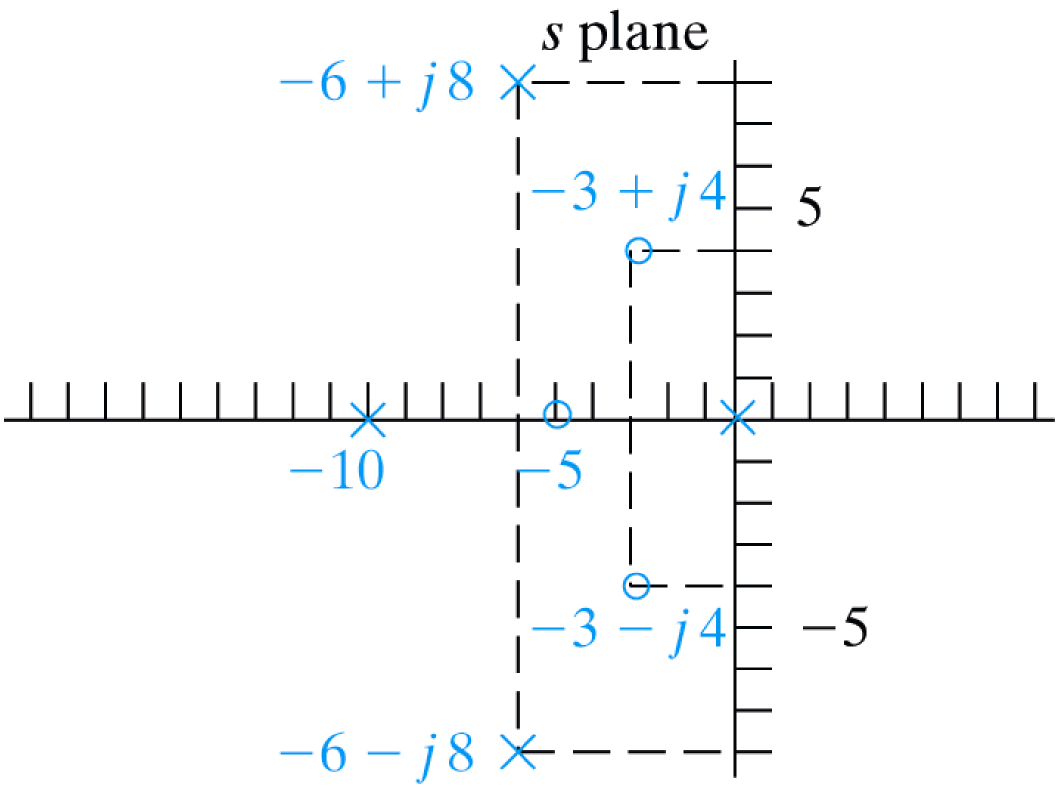
Section 12.8
Poles and Zeros of $F(s)$

Definition

- $F(s)$ can be expressed as the ratio of two factored polynomials $N(s)/D(s)$.
- The roots of the denominator $D(s)$ are called poles and are plotted as Xs on the complex s-plane.
- The roots of the numerator $N(s)$ are called zeros and are plotted as os on the complex s-plane.

Example

$$F(s)=\frac{N(s)}{D(s)}=\frac{(s+5)[s-(-3+j4)][s-(-3-j4)]}{s(s+10)[s-(-6+j8)][s-(-6-j8)]}.$$



Key points

- What is the definition of the Laplace transform?
- What are the Laplace transforms of unit step, impulse, exponential, and sinusoidal functions?
- What are the Laplace transforms of the derivative, integral, shift, and scaling of a function?
- How to perform partial fraction expansion for a rational function $F(s)$ and perform the inverse Laplace transform?