Chapter 7
Response of First-order RL and RC Circuits

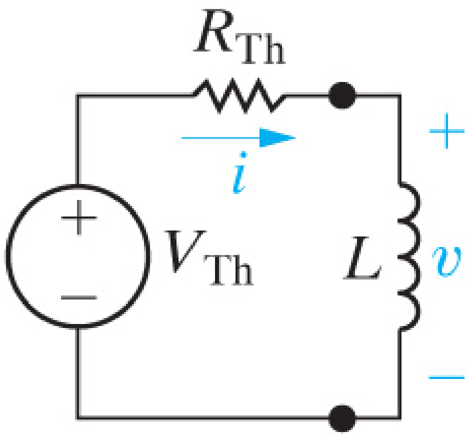
- 7.1-2 The Natural Response of *RL* and *RC* Circuits
- 7.3 The Step Response of *RL* and *RC* Circuits
- *7.4 A General Solution for Step and Natural Responses*
- 7.5 Sequential Switching
- 7.6 Unbounded Response

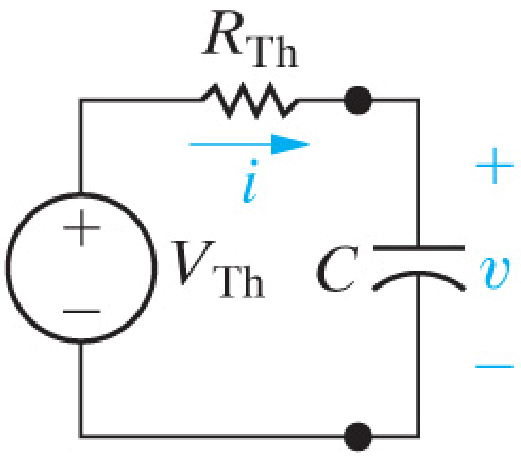
Overview

- Ch9-10 discuss “steady-state response” of linear circuits to “sinusoidal sources”. The math treatment is the same as the “dc response” except for introducing “phasors” and “impedances” in the algebraic equations.
- From now on, we will discuss “transient response” of linear circuits to “step sources” (Ch7-8) and general “time-varying sources” (Ch12-13). The math treatment involves with differential equations and Laplace transform.

First-order circuit

- A circuit that can be simplified to a Thévenin (or Norton) equivalent connected to either a single equivalent inductor or capacitor.





- In Ch7, the source is either none (natural response) or step source.

Key points

- Why an RC or RL circuit is charged or discharged as an exponential function of time?
- Why the charging and discharging speed of an RC or RL circuit is determined by RC or L/R?
- What could happen when an energy-storing element (C or L) is connected to a circuit with dependent source?

Section 7.1, 7.2
The Natural Response of RL and RC Circuits

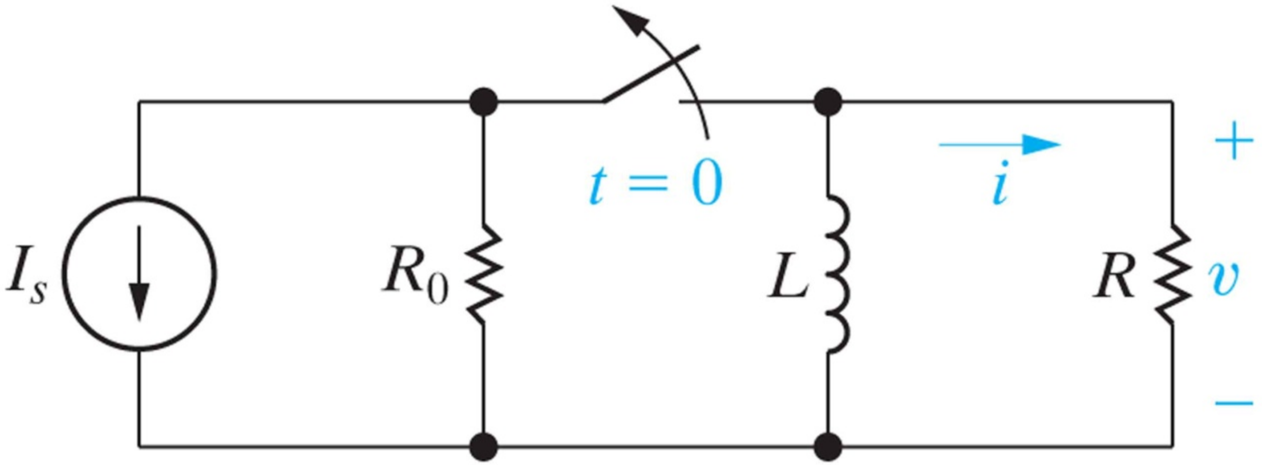
1. Differential equation & solution of a discharging RL circuit
2. Time constant
3. Discharging RC circuit

What is natural response?

- It describes the "discharging" of inductors or capacitors via a circuit of no dependent source.
- No external source is involved, thus termed as "natural" response.
- The effect will vanish as $t \to \infty$. The interval within which the natural response matters depends on the element parameters.

Circuit model of a discharging RL circuit

- Consider the following circuit model:



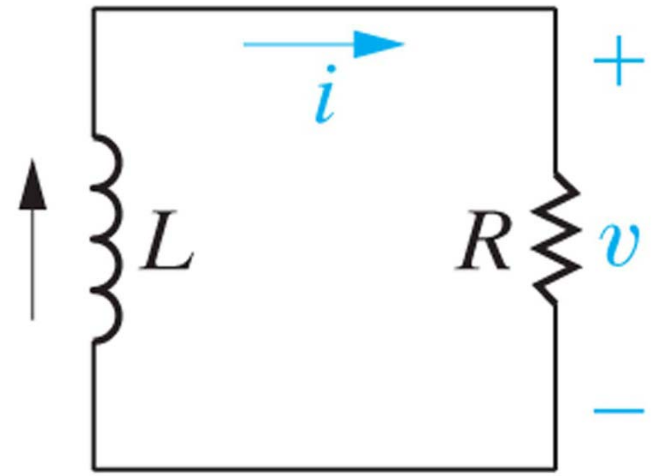
- For $t < 0$, the inductor L is short and carries a current I_s, while R_0 and R carry no current.
- For $t > 0$, the inductor current decreases and the energy is dissipated via R.

Ordinary differential equation (ODE), initial condition (IC)

- For $t > 0$, the circuit reduces to:

IC depends on initial energy of the inductor:

$$i(0^+) = i(0^-) \equiv I_0 = I_s$$



- By KVL, we got a first-order ODE for $i(t)$:

$$L\frac{d}{dt}i(t) + Ri(t) = 0.$$

where L, R are independent of both i, and t.

Solving the loop current

$$\text{ODE}: L\frac{d}{dt}i(t)+Ri(t)=0, \qquad \text{IC}: i(0^{+})\equiv I_0=I_s;$$

$$\Rightarrow L(di)+Ri(dt)=0,\ \frac{di}{i}=-\frac{R}{L}dt,$$

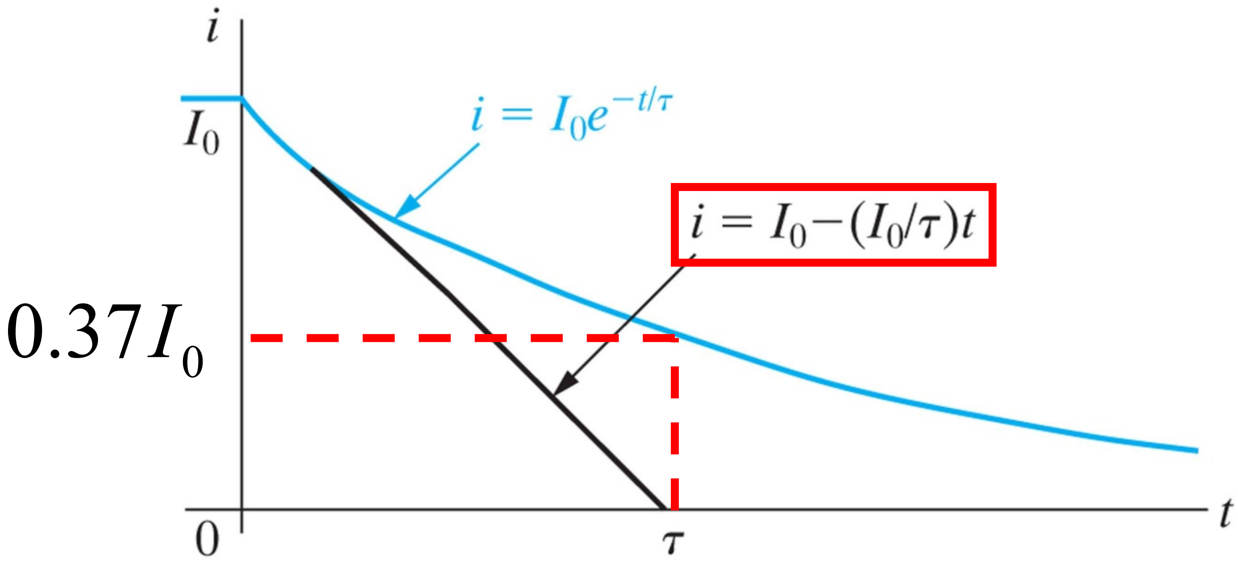
$$\Rightarrow \int_{i(0)}^{i(t)}\frac{di'}{i'}=-\frac{R}{L}\int_0^t dt',\ \ln i'\Big|_{i(0)}^{i(t)}=-\frac{R}{L}t'\Big|_0^t,$$

$$\Rightarrow \ln i(t)-\ln i(0)=\ln\frac{i(t)}{I_0}=-\frac{R}{L}t,$$

$\Rightarrow$ $i(t)=I_0e^{-(t/\tau)}$, where $\tau=\dfrac{L}{R}$ …time constant

Time constant describes the discharging speed

- The loop current $i(t)$ will drop to e^{-1} (≈37%) of its initial value I_0 within one time constant τ. It will be $<0.01I_0$ after elapsing 5τ.
- If $i(t)$ is approximated by a linear function, it will vanish in one time constant.



Solutions of the voltage, power, and energy

- The voltage across R (or L) is:

$$v(t) = Ri(t) = \begin{cases} 0, \ \text{for } t \le 0^-, \\ RI_0 e^{-(t/\tau)}, \ \text{for } t \ge 0^+. \end{cases}$$ …abrupt change at $t=0$.

- The instantaneous power dissipated in R is:

$$p(t) = i^2(t)R = I_0^2 R e^{-(2t/\tau)}, \ \text{for } t \ge 0^+.$$

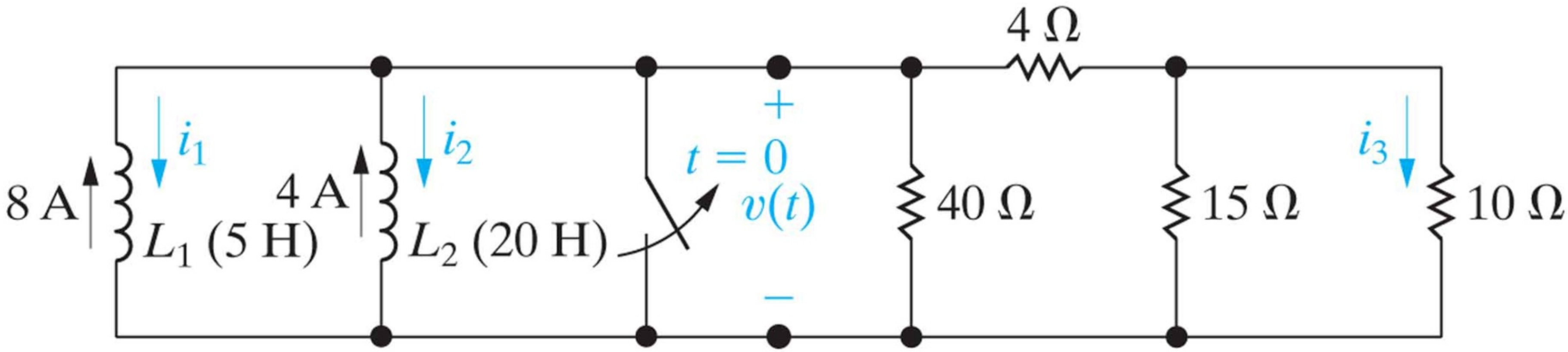
- The energy dissipated in the resistor R is:

$$w = \int_0^t p(t')dt' = I_0^2 R \int_0^t e^{-(2t'/\tau)} dt'$$
$$= w_0\left[1 - e^{-(2t/\tau)}\right], \ \boxed{w_0 = LI_0^2/2}, \ \text{for } t \ge 0^+.$$

initial energy stored in L

Example 7.2: Discharge of parallel inductors (1)

- Q: Find $i_1(t)$, $i_2(t)$, $i_3(t)$, and the energies w_1, w_2 stored in L_1, L_2 in steady state $(t \to \infty)$.



- For $t < 0$: (1) L_1, L_2 are short, and (2) no current flows through any of the 4 resistors, $\Rightarrow$

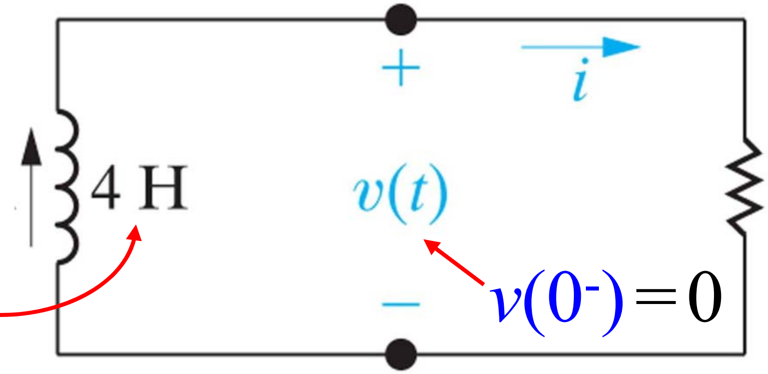
$$i_1(0^-) = -8\text{ A},\ \ i_2(0^-) = -4\text{ A},\ \ i_3(0^-) = 0,$$

$$w_1(0^-) = (5\text{ H})(8\text{ A})^2/2 = 160\text{ J}, w_2(0^-) = (20)(4)^2/2 = 160\text{ J}.$$

Example 7.2: Solving the equivalent RL circuit (2)

- For $t > 0$, switch is open, the initial energy stored in the 2 inductors is dissipated via the 4 resistors. The equivalent circuit becomes:

$I_0 = (8+4) = 12$ A



$R_{eq} =$ 8 Ω

$v(0^-) = 0$

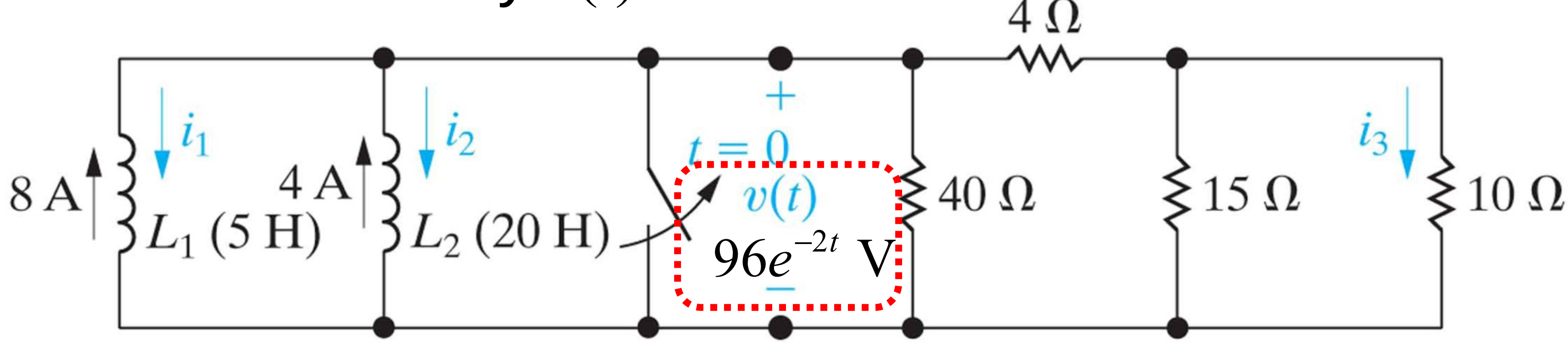
$L_{eq} = (5\text{H}//20\text{H}) = 4$ H

- The solutions to $i(t)$, $v(t)$ are:

$$\tau = \frac{L_{eq}}{R_{eq}} = \frac{4}{8} = 0.5 \text{ s}, \Rightarrow \begin{cases} i(t) = I_0 e^{-(t/\tau)} = 12e^{-2t} \text{ A}, \\ v(t) = Ri(t) = 96e^{-2t} \text{ V}. \end{cases}$$

Example 7.2: Solving the inductor currents (3)

- The two inductor currents $i_1(t)$, $i_2(t)$ can be calculated by $v(t)$:



$$i_1(t) = i_1(0) + \frac{1}{L_1}\int_0^t v(t')dt' = -8 + \frac{1}{5}\int_0^t 96e^{-2t}dt' = 1.6 - 9.6e^{-2t} \text{ A.}$$

$$i_2(t) = i_2(0) + \frac{1}{L_2}\int_0^t v(t')dt' = -4 + \frac{1}{20}\int_0^t 96e^{-2t}dt' = -1.6 - 2.4e^{-2t} \text{ A.}$$

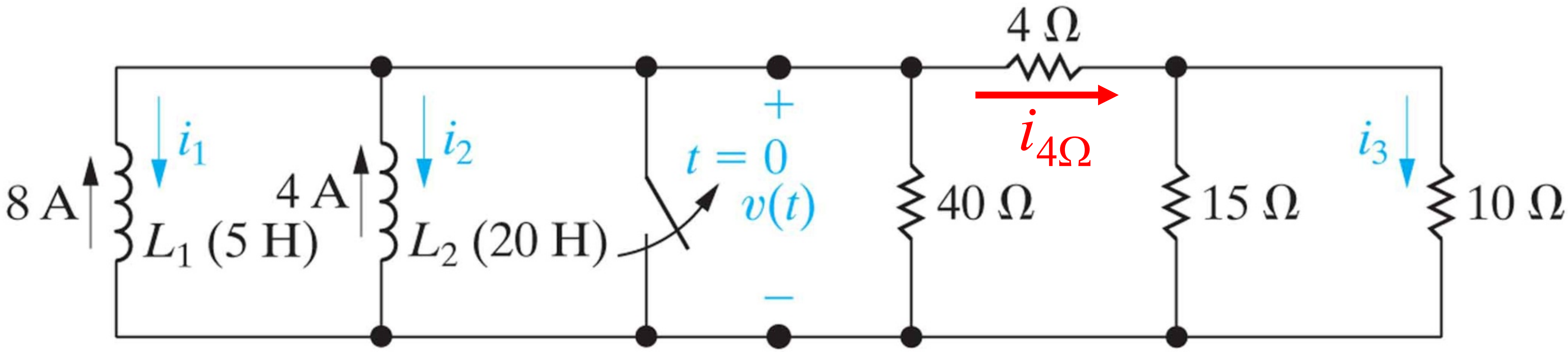
Example 7.2: Solutions in steady state (4)

- Since
$$i_1(t) = 1.6 - 9.6e^{-2t} \rightarrow 1.6\ \mathrm{A},$$
$$i_2(t) = -1.6 - 2.4e^{-2t} \rightarrow -1.6\ \mathrm{A},$$
the two inductors form a closed current loop!

- The energies stored in the two inductors are:
$$w_1(t \rightarrow \infty) = \frac{1}{2}(5\ \mathrm{H})(1.6\ \mathrm{A})^2 = 6.4\ \mathrm{J},$$
$$w_2(t \rightarrow \infty) = \frac{1}{2}(20)(-1.6)^2 = 25.6\ \mathrm{J},$$
which is ~10% of the initial energy in total.

Example 7.2: Solving the resistor current (5)

- By current division, $i_3(t) = 0.6 i_{4\Omega}(t)$, while $i_{4\Omega}(t)$ can be calculated by $v(t)$:

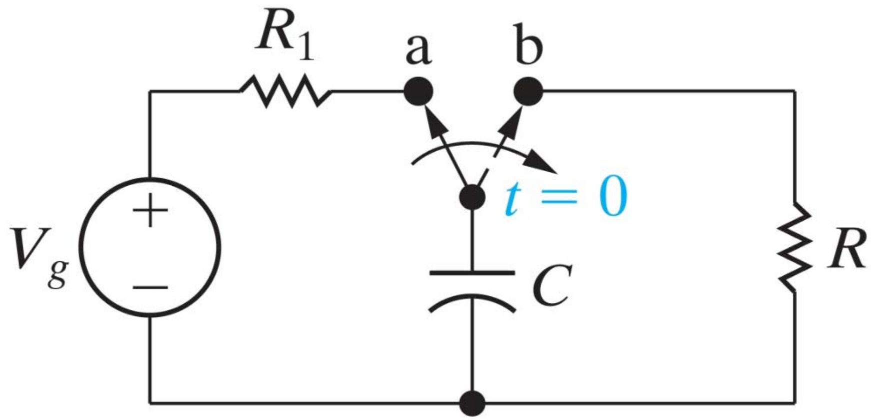


$$i_{4\Omega}(t) = \frac{v(t)}{4\Omega + (15\Omega // 10\Omega)} = \frac{96e^{-2t}}{10} = 9.6e^{-2t} \text{ A.}$$

$$\Rightarrow i_3(t) = \frac{3}{5} i_{4\Omega}(t) = 5.76e^{-2t} \text{ A.}$$

Circuit model of a discharging RC circuit

- Consider the following circuit model:



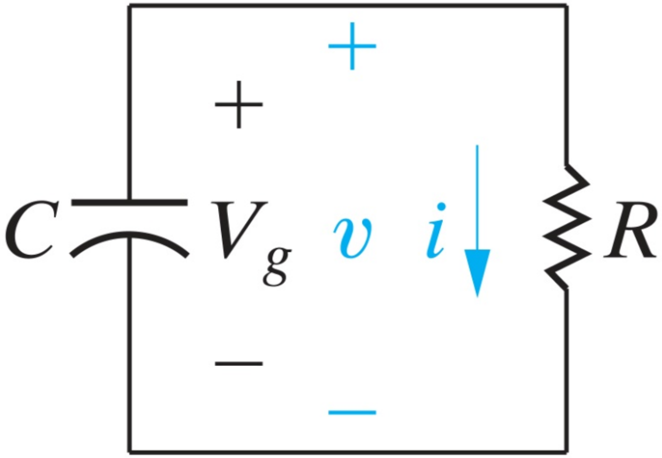
- For $t < 0$, C is open and biased by a voltage V_g, while R_1 and R carry no current.
- For $t > 0$, the capacitor voltage decreases and the energy is dissipated via R.

ODE & IC

- For $t > 0$, the circuit reduces to:

  IC depends on initial energy of the capacitor:

  $$v(0^+) = v(0^-) \equiv V_0 = V_g$$



- By KCL, we got a first-order ODE for $v(t)$:

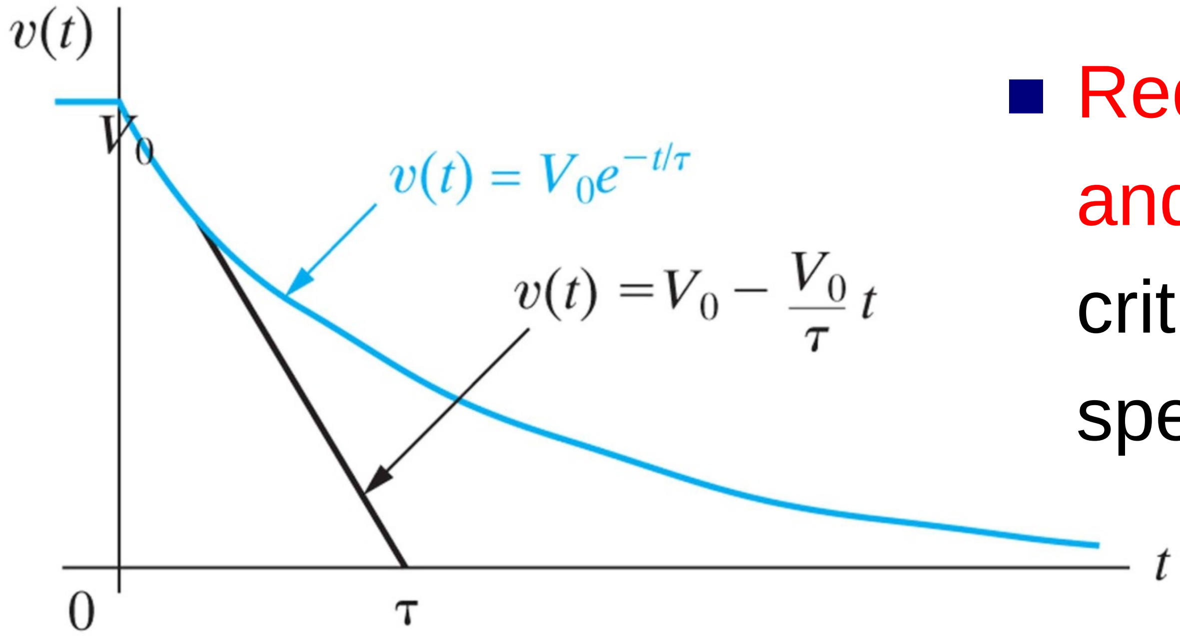
$$C\frac{d}{dt}v(t) + \frac{v(t)}{R} = 0.$$

where C, R are independent of both v, and t.

Solving the parallel voltage

$$\text{ODE}: C\frac{d}{dt}v(t)+\frac{v(t)}{R}=0, \qquad \text{IC}: v(0^+)\equiv V_0=V_g;$$

$\Rightarrow v(t)=V_0e^{-(t/\tau)}$, where $\tau = RC$ …time constant



- Reducing R (loss) and parasitic C is critical for high-speed circuits.

Solutions of the current, power, and energy

- The loop current is:

$$i(t)=\frac{v(t)}{R}=\begin{cases}0, & \text{for } t\le 0^-,\\ (V_0/R)e^{-(t/\tau)}, & \text{for } t\ge 0^+.\end{cases}$$

...abrupt change at $t=0$.

- The instantaneous power dissipated in R is:

$$p(t)=\frac{v^2(t)}{R}=\frac{V_0^2}{R}e^{-(2t/\tau)},\ \text{for } t\ge 0^+.$$

- The energy dissipated in the resistor R is:

$$w=\int_0^t p(t')dt'=w_0\left[1-e^{-(2t/\tau)}\right],\ \boxed{w_0=\frac{CV_0^2}{2}},\ \text{for } t\ge 0^+.$$

initial energy stored in C

Procedures to get natural response of RL, RC circuits

1. Find the equivalent circuit.
2. Find the initial conditions: initial current I_0 through the equivalent inductor, or initial voltage V_0 across the equivalent capacitor.
3. Find the time constant of the circuit by the values of the equivalent R, L, C:

$$\tau = L/R, \text{ or } RC;$$

4. Directly write down the solutions:

$$i(t) = I_0 e^{-(t/\tau)}, \; v(t) = V_0 e^{-(t/\tau)}.$$

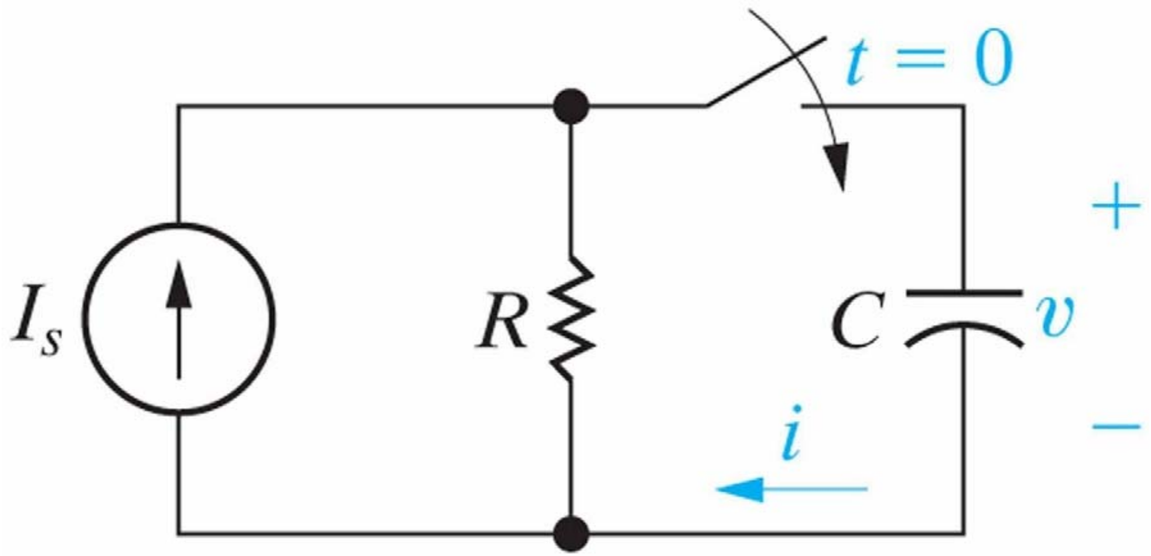
Section 7.3
The Step Response of RL and RC Circuits

1. Charging an RC circuit
2. Charging an RL circuit

What is step response?

- The response of a circuit to the sudden application of a constant voltage or current source, describing the charging behavior of the circuit.
- Step (charging) response and natural (discharging) response show how the signal in a digital circuit switches between Low and High with time.

ODE and IC of a charging RC circuit



IC depends on initial energy of the capacitor:

$$v(0^+) = v(0^-) \equiv V_0$$

- Derive the governing ODE by KCL:

$$I_s = \frac{v(t)}{R} + C\frac{d}{dt}v(t), \Rightarrow \frac{dv}{dt} = -\frac{1}{\tau}\left(v - V_f\right),$$

where $\tau = RC$, $V_f = I_s R$ are the time constant and final (steady-state) parallel voltage.

Solving the parallel voltage, branch currents

$$\frac{dv}{dt} = -\frac{1}{\tau}\left(v - V_f\right), \Rightarrow \frac{dv}{v - V_f} = -\frac{1}{\tau}dt,$$

$$\int_{V_0}^{v(t)} \frac{dv'}{v' - I_s R} = -\frac{1}{\tau}\int_0^t dt', \quad \ln\frac{v(t) - V_f}{V_0 - V_f} = -\frac{t}{\tau}, \quad \frac{v(t) - V_f}{V_0 - V_f} = e^{-t/\tau},$$

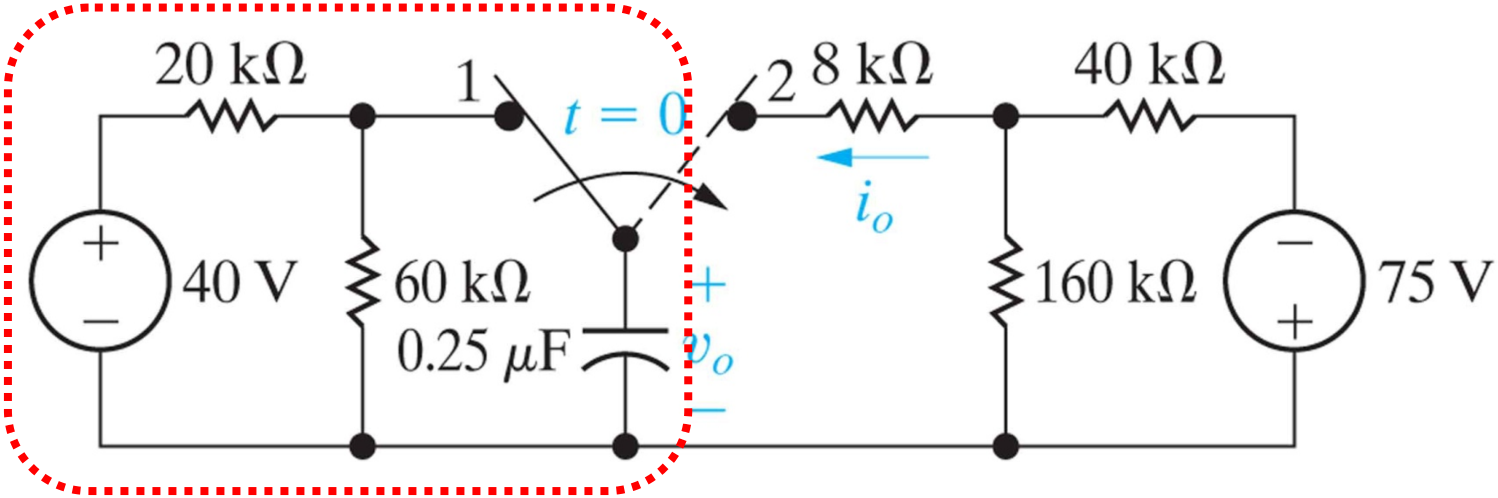
$$\Rightarrow v(t) = V_f + \left(V_0 - V_f\right)e^{-t/\tau}.$$

- The charging and discharging processes have the same speed (same time constant $\tau = RC$).
- The branch currents through C and R are:

$$i_C(t) = C\frac{d}{dt}v(t) = \left(I_s - \frac{V_0}{R}\right)e^{-t/\tau}, i_R(t) = \frac{v(t)}{R}, \text{ for } t > 0.$$

Example 7.6 (1)

- Q: Find $v_o(t)$, $i_o(t)$ for $t \geq 0$.

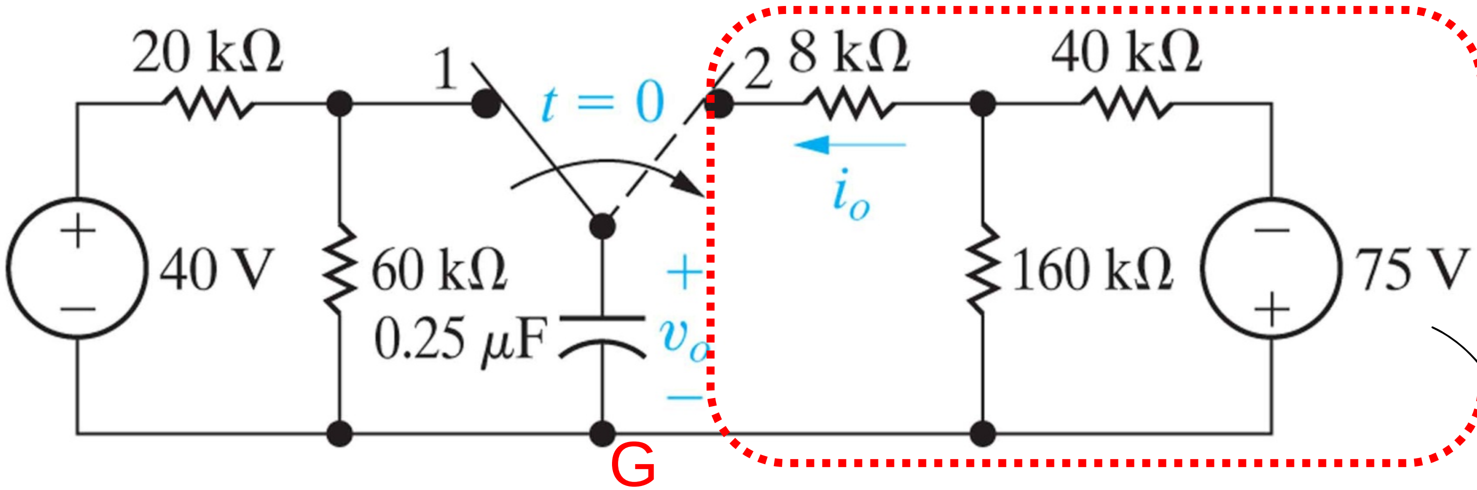


- For $t < 0$, the switch is connected to Terminal 1 for long, the capacitor is an open circuit:

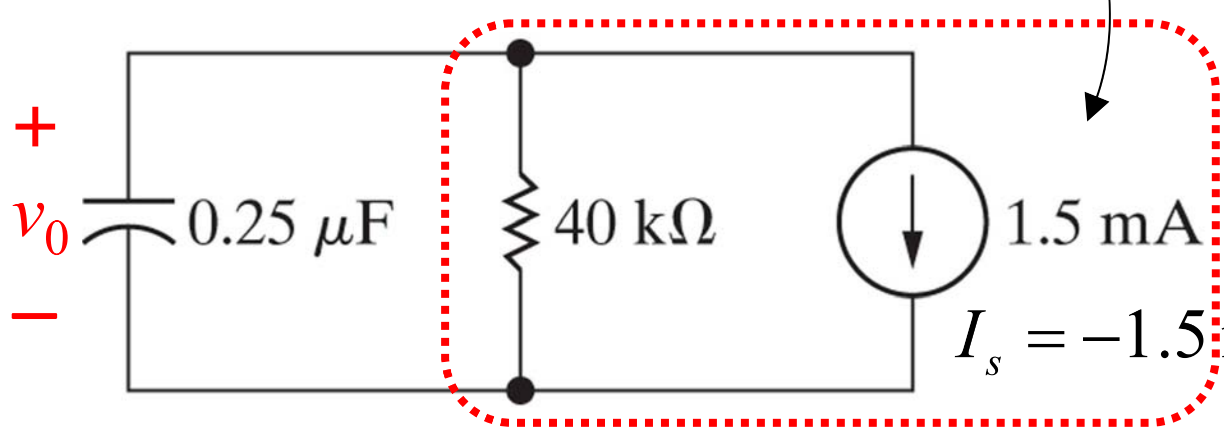
$$v_o(0^-) = (40\text{ V})\frac{60\text{ k}\Omega}{(20+60)\text{ k}\Omega} = 30\text{ V},\ i_o(0^-) = 0.$$

Example 7.6 (2)

- At $t \geq 0^+$, the "charging circuit" with two terminals 2 and G can be reduced to a Norton equivalent:



$$v_o(0^+) = v_o(0^-)$$
$$\equiv V_0 = 30 \text{ V}$$



$$I_s = -1.5 \text{ mA}$$

Example 7.6 (3)

- The time constant and the final capacitor voltage of the charging circuit are:

$$\tau = RC = (40\ \text{k}\Omega)(0.25\ \mu\text{F}) = 10\ \text{ms},$$

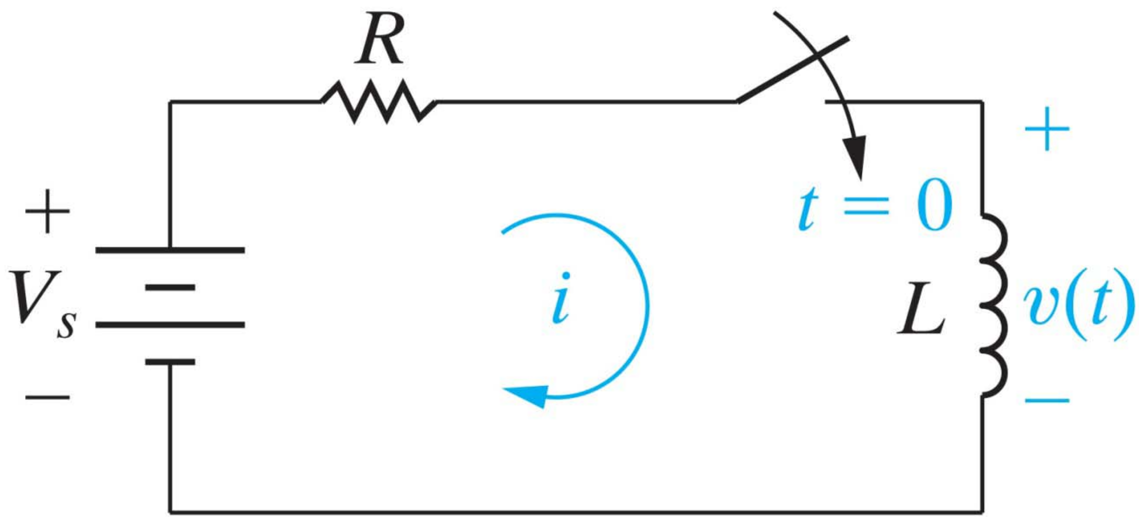
$$V_f = I_s R = (-1.5\ \text{mA})(40\ \text{k}\Omega) = -60\ \text{V}.$$

$$\Rightarrow v_o(t) = V_f + \left(V_0 - V_f\right)e^{-t/\tau} = -60 + 90e^{-100t}\ \text{V}.$$

$$\Rightarrow i_o(t) = \left(I_s - V_0/R\right)e^{-t/\tau} = -2.25e^{-100t}\ \text{mA}.$$

- At the time of switching, the capacitor voltage is continuous: $v_o(0^-) = 30\ \text{V} = v_0(0^+) = (-60+90)\ \text{V}$, while the current i_o jumps from 0 to $-2.25\ \text{mA}$.

Charging an RL circuit



IC depends on initial energy of the inductor:

$$i(0^+) = i(0^-) \equiv I_0$$

$$V_s = Ri(t) + L\frac{d}{dt}i(t), \Rightarrow i(t) = I_f + \left(I_0 - I_f\right)e^{-t/\tau},$$

where $\tau = \dfrac{L}{R}$, $I_f = \dfrac{V_s}{R}$.

- The charging and discharging processes have the same speed (same time constant $\tau = L/R$).

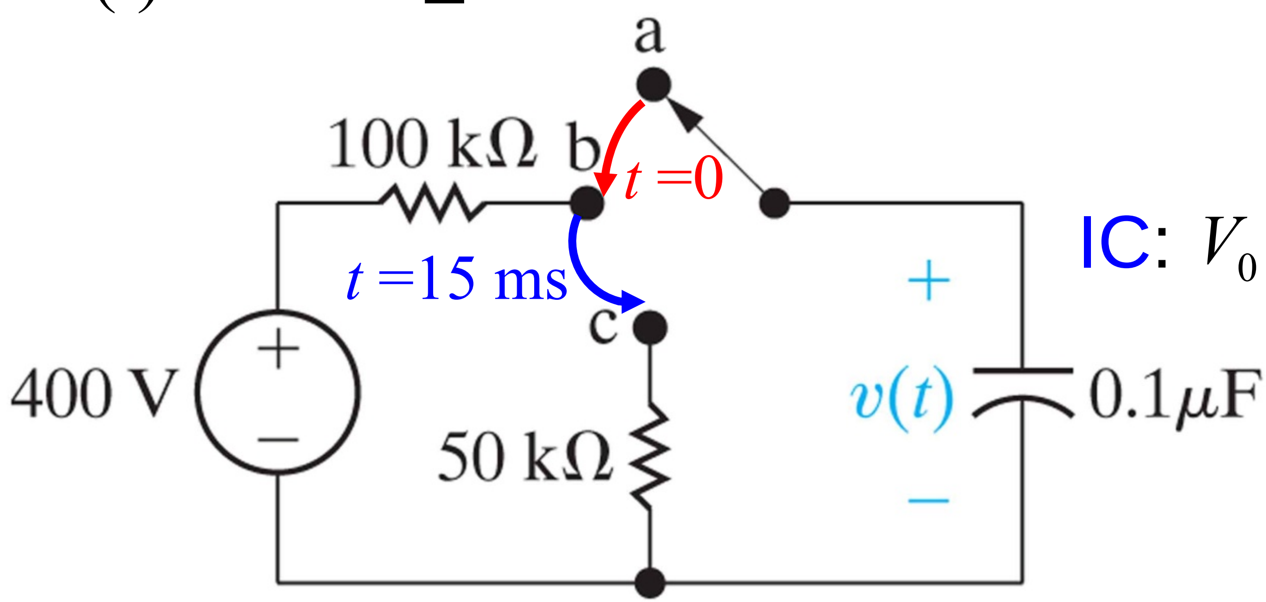
Section 7.5
Sequential Switching

What is and how to solve sequential switching?

- Sequential switching means switching occurs n (≥ 2) times.
- It is analyzed by dividing the process into $n+1$ time intervals. Each of them corresponds to a specific circuit.
- Initial conditions of a particular interval are determined from the solution of the preceding interval. Inductive currents and capacitive voltages are particularly important for they cannot change abruptly.
- Laplace transform (Ch12) can solve it easily.

Example 7.12: Charging and discharging a capacitor (1)

- Q: $v(t)=?$ for $t \geq 0$.



IC: $V_0 = v(0^{\pm}) = 0$

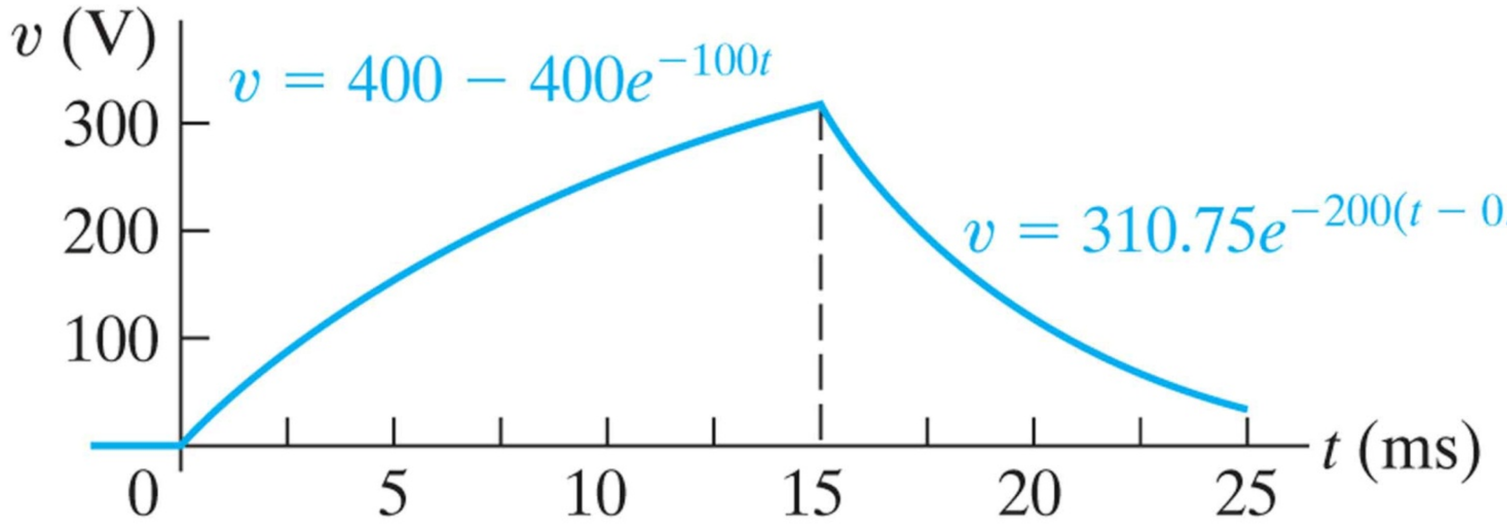
- For $0 \leq t \leq 15$ ms, the 400-V source charges the capacitor via the 100-kΩ resistor, $\Rightarrow V_f = 400$ V, $\tau = RC = 10$ ms, and the capacitor voltage is:

$$v_1(t) = V_f + (V_0 - V_f)e^{-t/\tau} = 400 - 400e^{-100t} \text{ V}.$$

Example 7.12 (2)

- For $t > 15$ ms, the capacitor is disconnected from the 400-V source and is discharged via the 50-kΩ resistor, $\Rightarrow V_0 = v_1(15 \text{ ms}) = 310.75$ V, $V_f = 0$, $\tau = RC = 5$ ms. The capacitor voltage is:

$$v_2(t) = V_0 e^{-\underline{(t-t_0)}/\tau} = 310.75 e^{-200(t-0.015)} \text{ V}.$$



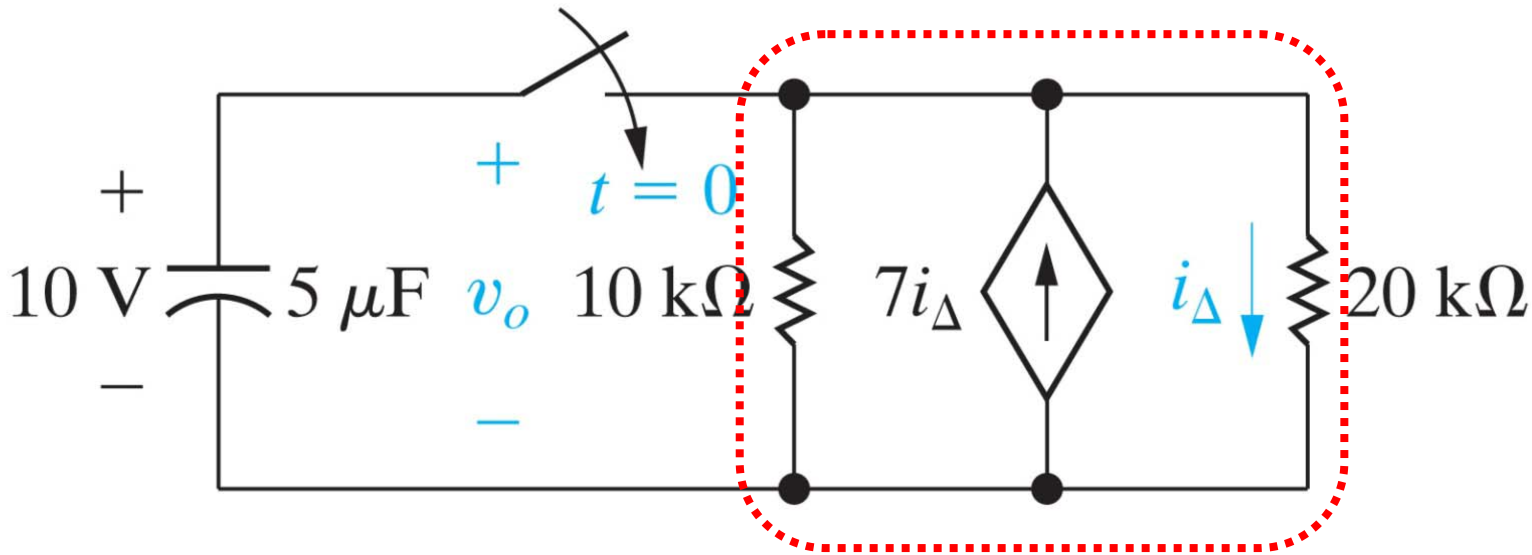
Section 7.6
Unbounded Response

Definition and reason of unbounded response

- An unbounded response means the voltages or currents increase with time without limit.
- It occurs when the Thévenin resistance is negative ($R_{Th} < 0$), which is possible when the first-order circuit contains dependent sources.

Example 7.13: (1)

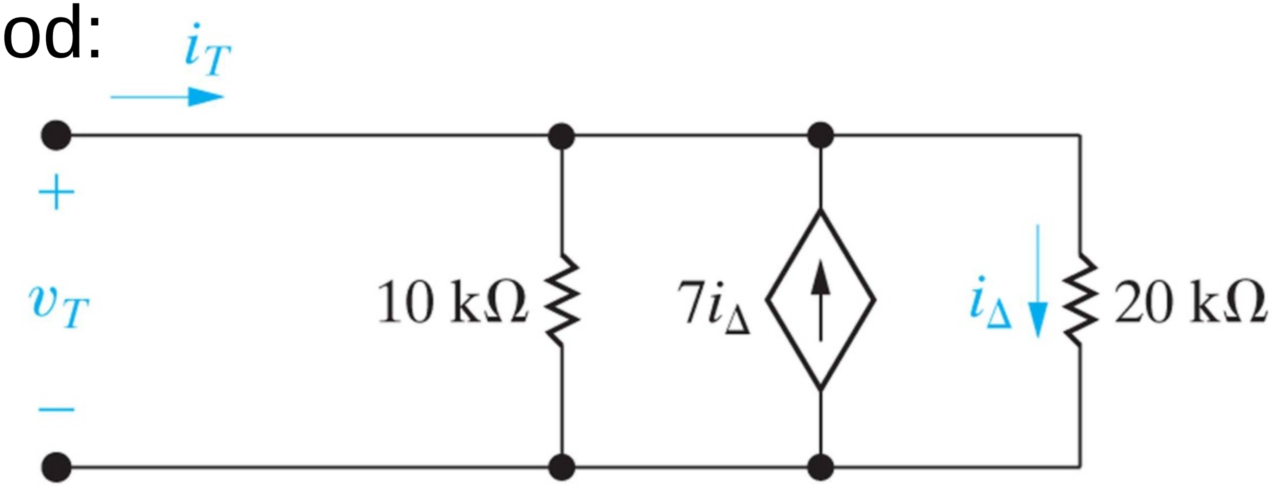
- Q: $v_o(t) = ?$ for $t \geq 0$.



- For $t > 0$, the capacitor seems to “discharge” (not really, to be discussed) via a circuit with a current-controlled current source, which can be represented by a Thévenin equivalent.

Example 7.13 (2)

- Since there is no independent source, $V_{Th}=0$, while R_{Th} can be determined by the test source method:

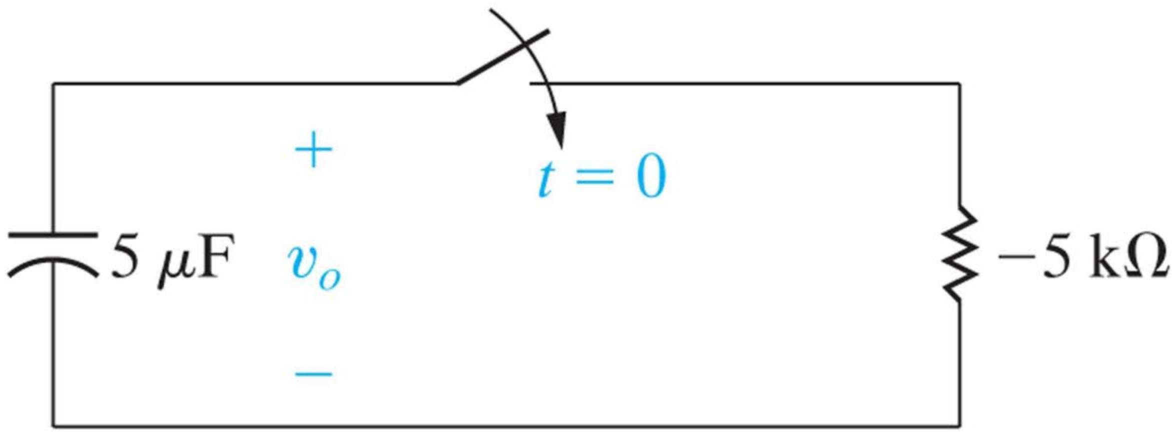


$$i_T = \frac{v_T}{10\text{ k}\Omega} + \frac{v_T}{20\text{ k}\Omega} - 7\frac{v_T}{20\text{ k}\Omega},$$

$$\Rightarrow R_{Th} = v_T / i_T = \text{-5 k}\Omega < 0.$$

Example 7.13 (3)

- For $t \geq 0$, the equivalent circuit and governing differential equation become:

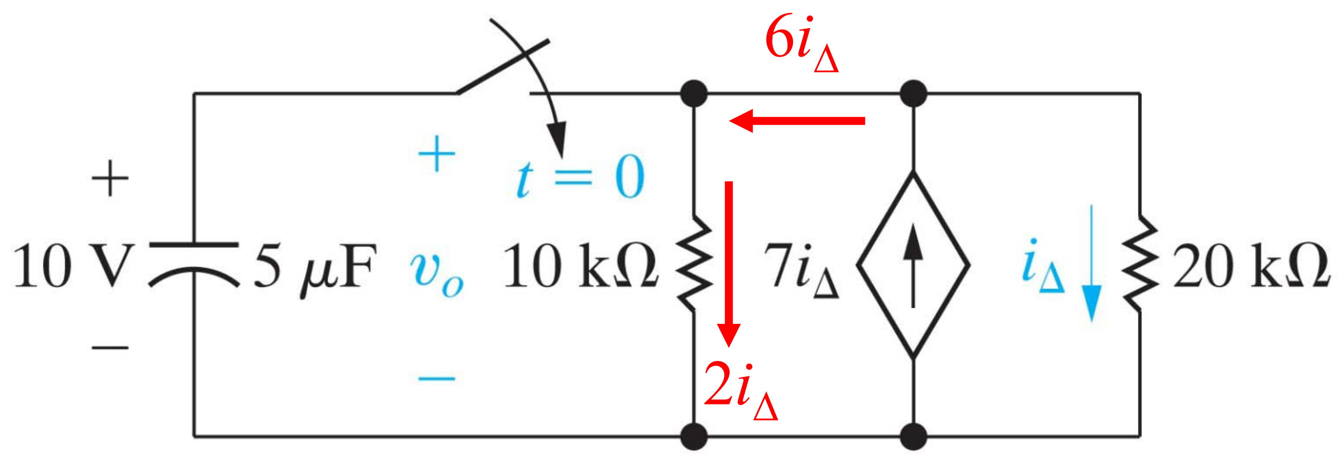


$$V_0 = v_o(0^+) = 10\text{ V},\ \tau = RC = -25\text{ ms} < 0,$$

$$\Rightarrow v_o(t) = V_0 e^{-t/\tau} = 10 \boxed{e^{+40t}}\text{ V}.$$ ..grow without limit.

Why the voltage is unbounded?

- Since $10\text{-k}\Omega$, $20\text{-k}\Omega$ resistors are in parallel, $\Rightarrow$ $i_{10\text{k}\Omega} = 2i_\Delta$, the capacitor is actually charged (not discharged) by a current of $4i_\Delta$!
- Charging effect will increase v_o, which will in turn increase the charging current ($i_\Delta = v_o/20\ \text{k}\Omega$) and v_o itself. The positive feedback makes v_o soaring.



Lesson for circuit designers & device fabrication engineers

- Undesired interconnection between a capacitor and a sub-circuit with dependent source (e.g. transistor) could be catastrophic!

Key points

- Why an RC or RL circuit is charged or discharged as an exponential function of time?
- Why the charging and discharging speed of an RC or RL circuit is determined by RC or L/R?
- What could happen when an energy-storing element (C or L) is connected to a circuit with dependent source?