Chapter 4 Techniques of Circuit Analysis

4.1	Terminology
4.2-4.4	The Node-Voltage Method (NVM)
4.5-4.7	The Mesh-Current Method (MCM)
4.8	Choosing NVM or MCM
4.9	Source Transformations
4.10-4.11	Thévenin and Norton Equivalents
4.12	Maximum Power Transfer
4.13	Superposition

Overview

- Circuit analysis by series-parallel reduction and Δ-Y transformations might be cumbersome or even impossible when the circuits are structurally complicated and/or involve with a lot of elements.
- Systematic methods that can describe circuits with minimum number of simultaneous equations are of high interest.

Key points

- How to solve a circuit by the Node-Voltage Method and Mesh-Current Method systematically?
- What is the meaning of equivalent circuit? Why is it useful?
- How to get the Thévenin equivalent circuit?
- What is superposition? Why is it useful?

Section 4.1
Terminology

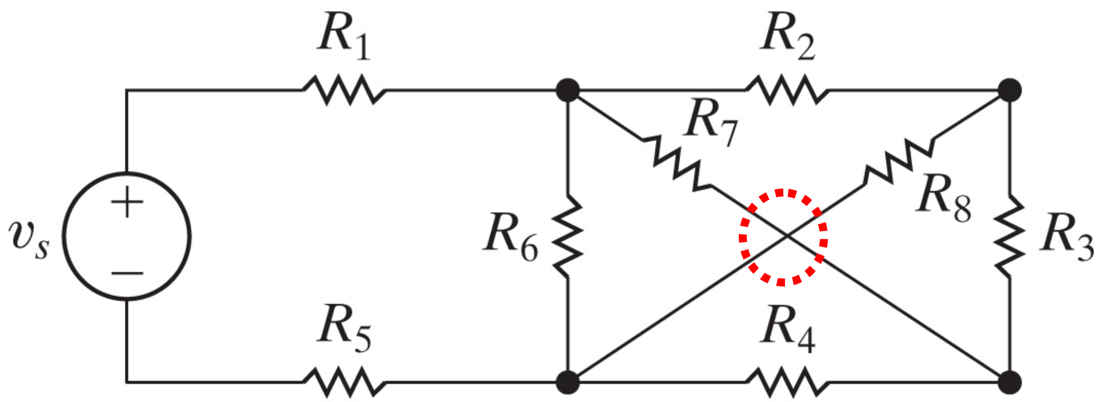
Definition

TABLE 4.1 Terms for Describing Circuits

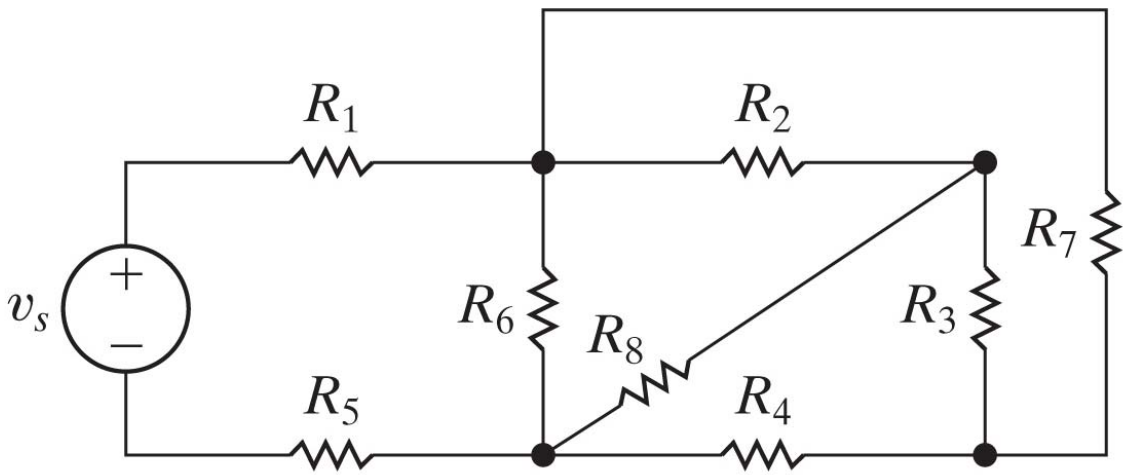
Name	Definition
node	A point where two or more circuit elements join
essential node	A node where three or more circuit elements join
path	A trace of adjoining basic elements with no elements included more than once
branch	A path that connects two nodes
essential branch	A path which connects two essential nodes without passing through an essential node
loop	A path whose last node is the same as the starting node
mesh	A loop that does not enclose any other loops
planar circuit	A circuit that can be drawn on a plane with no crossing branches

Planar circuits

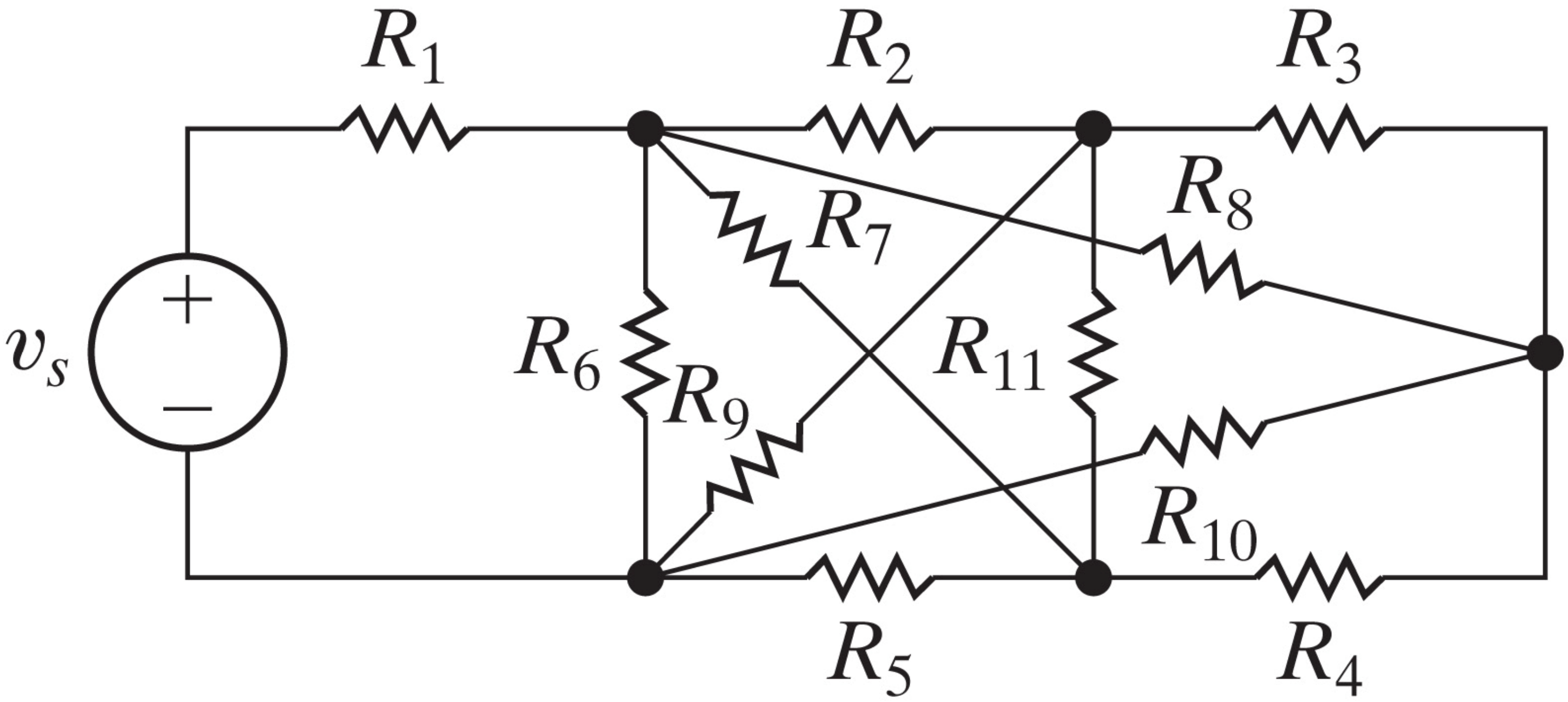
- Circuits without crossing branches.



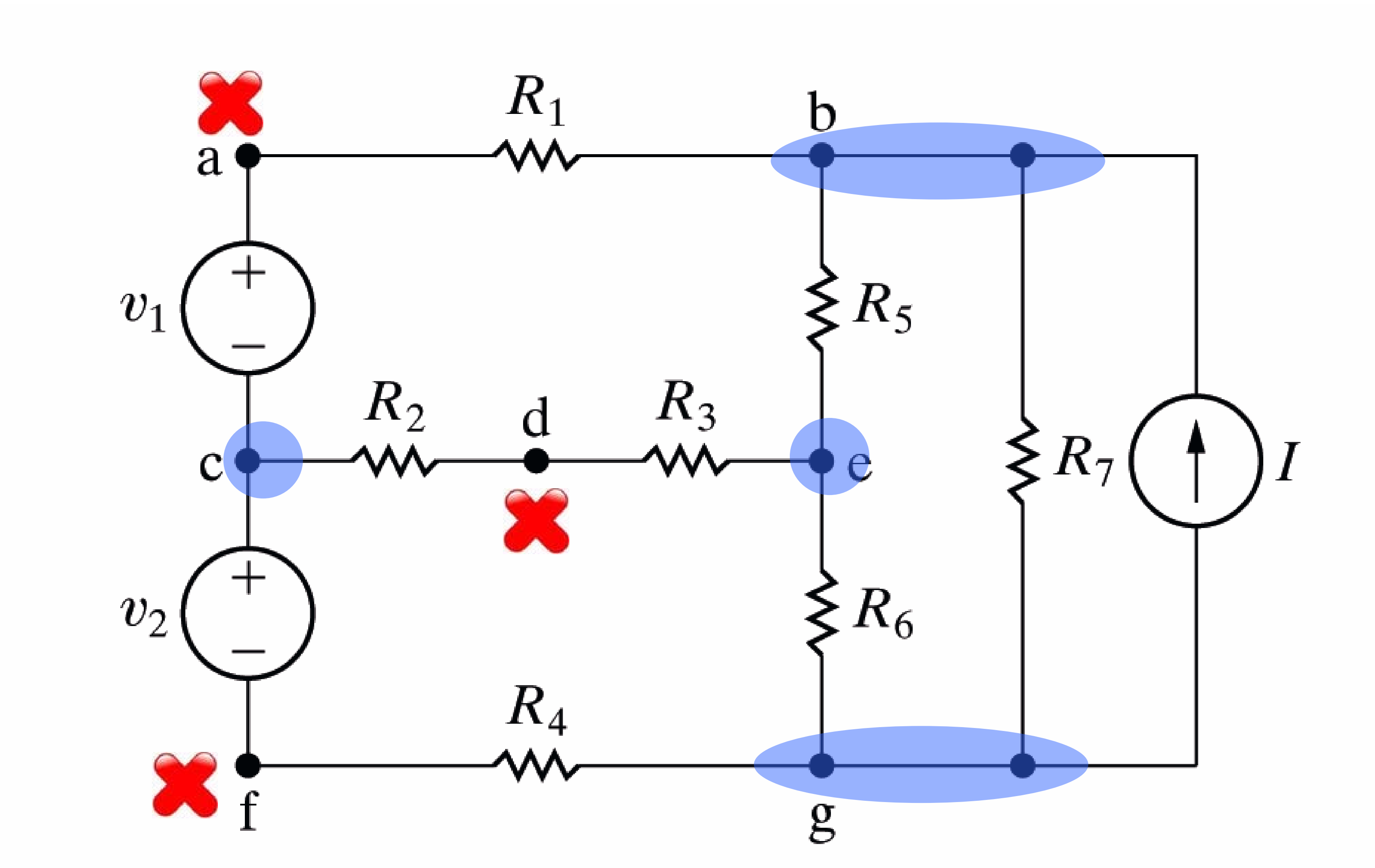
Not in contact



Example of a nonplanar circuit

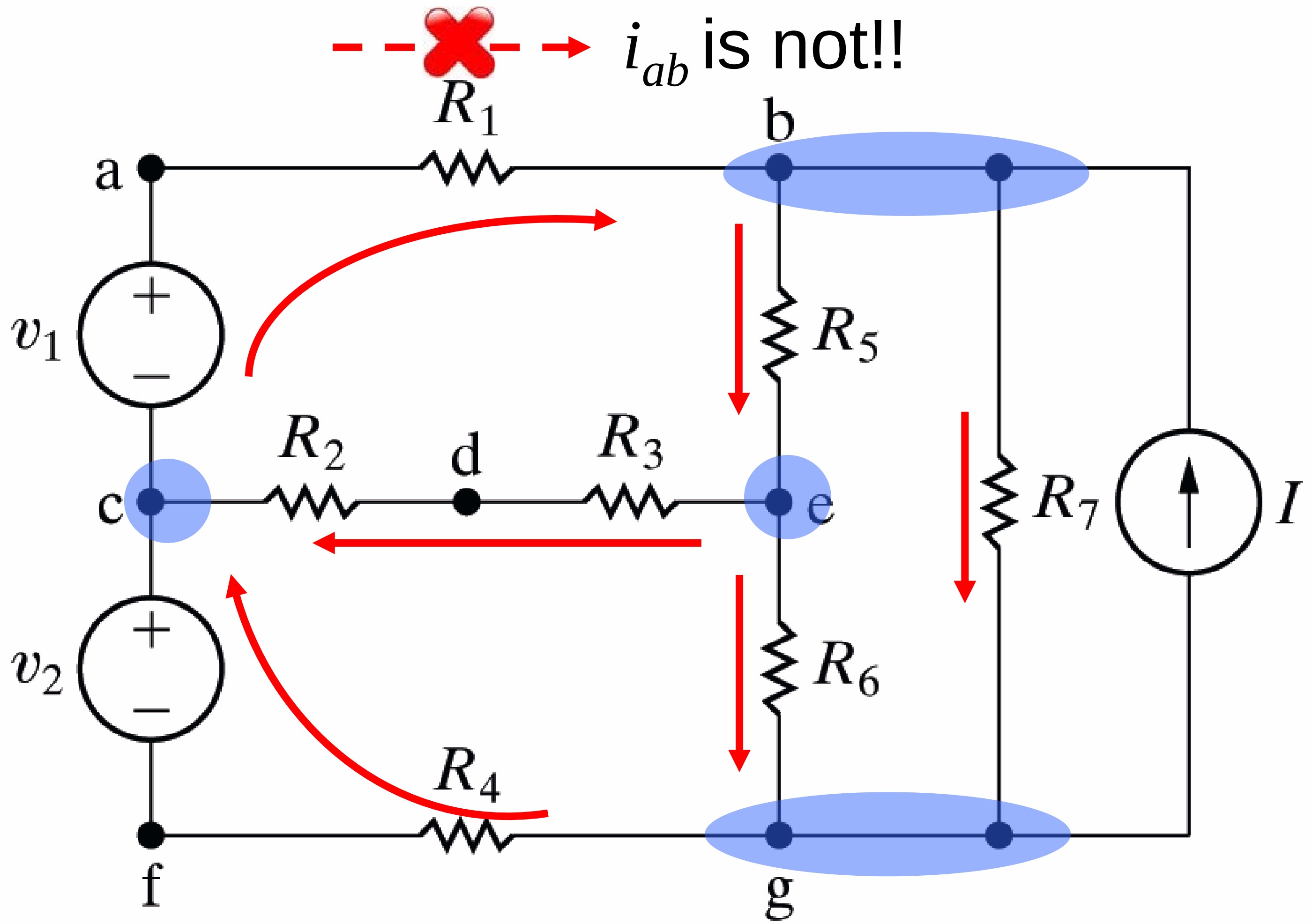


Identifying essential nodes in a circuit



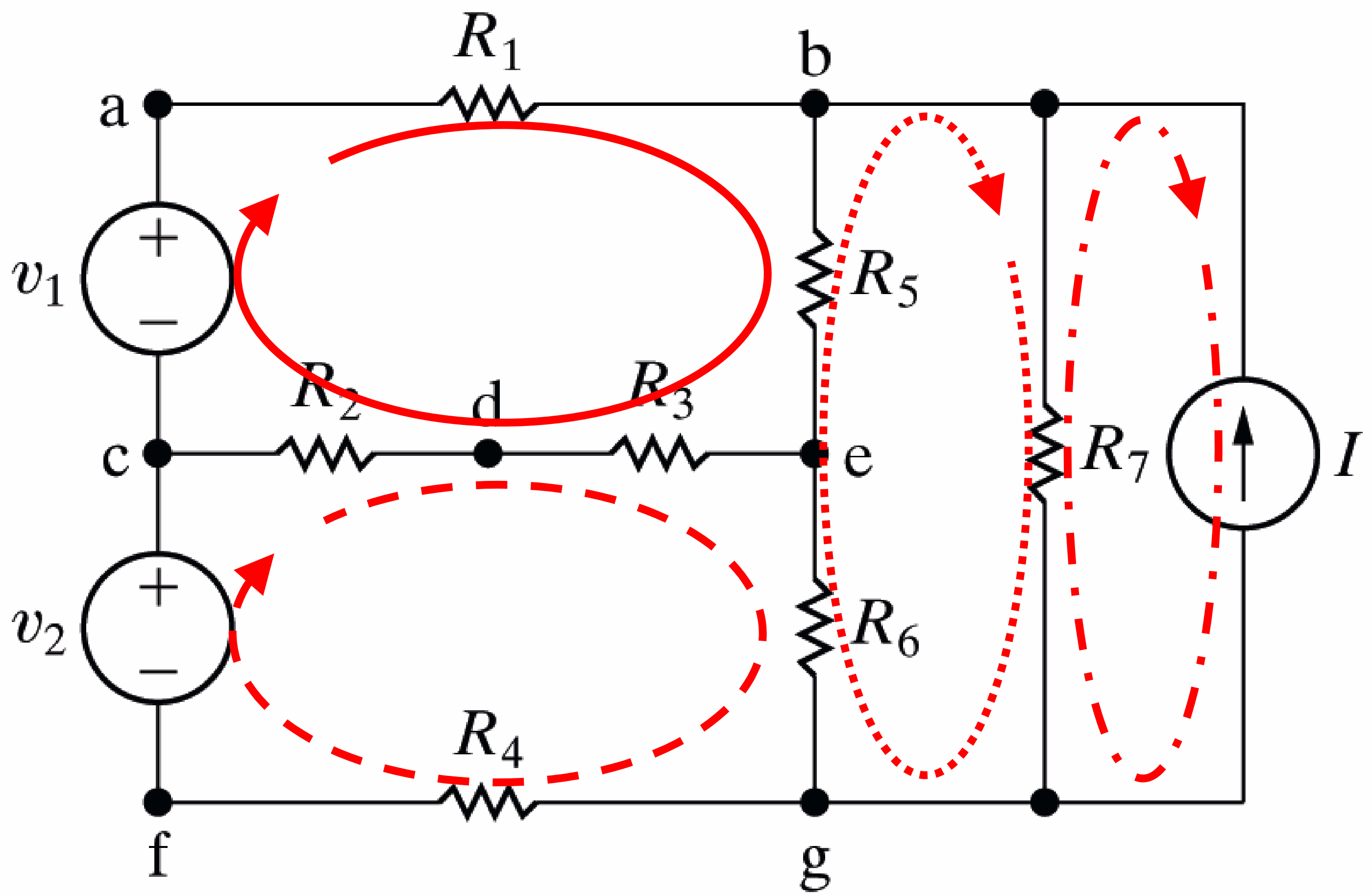
- Number of essential nodes is denoted by n_e.

Identifying essential branches in a circuit



- Number of essential branches is denoted by b_e.

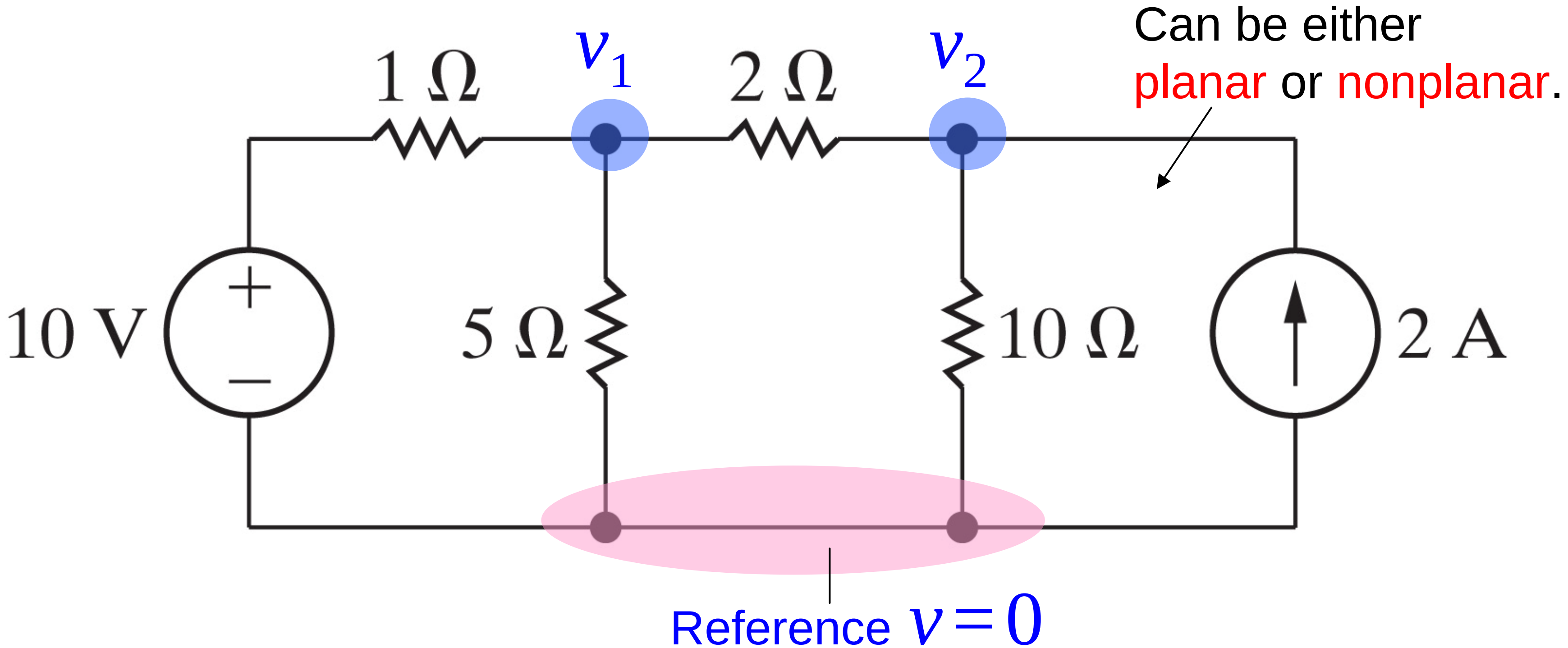
Identifying meshes in a circuit



Section 4.2-4.4 The Node-Voltage Method (NVM)

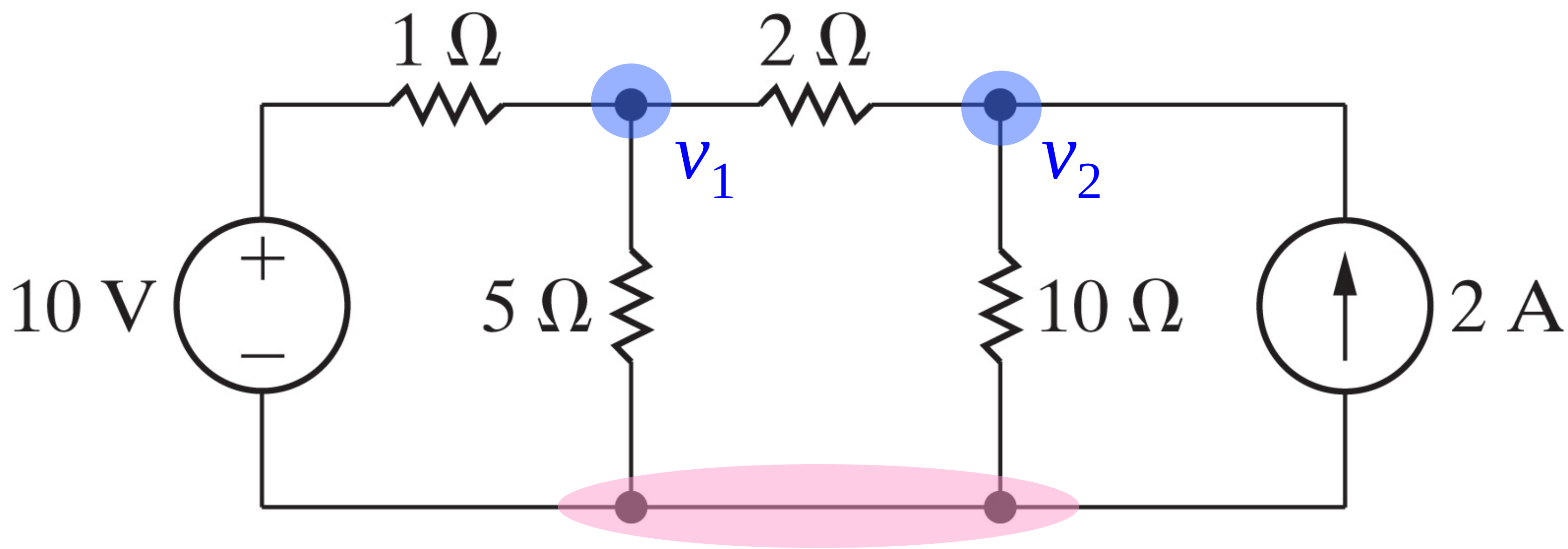
1. Standard procedures
2. Use of supernode

Step 1: Select one of the n_e essential nodes as the reference node



- Selection is arbitrary. Usually, the node connecting to the most branches is selected to simplify the formulation.

Step 2: List n_e-1 equations by KCL, Solve them



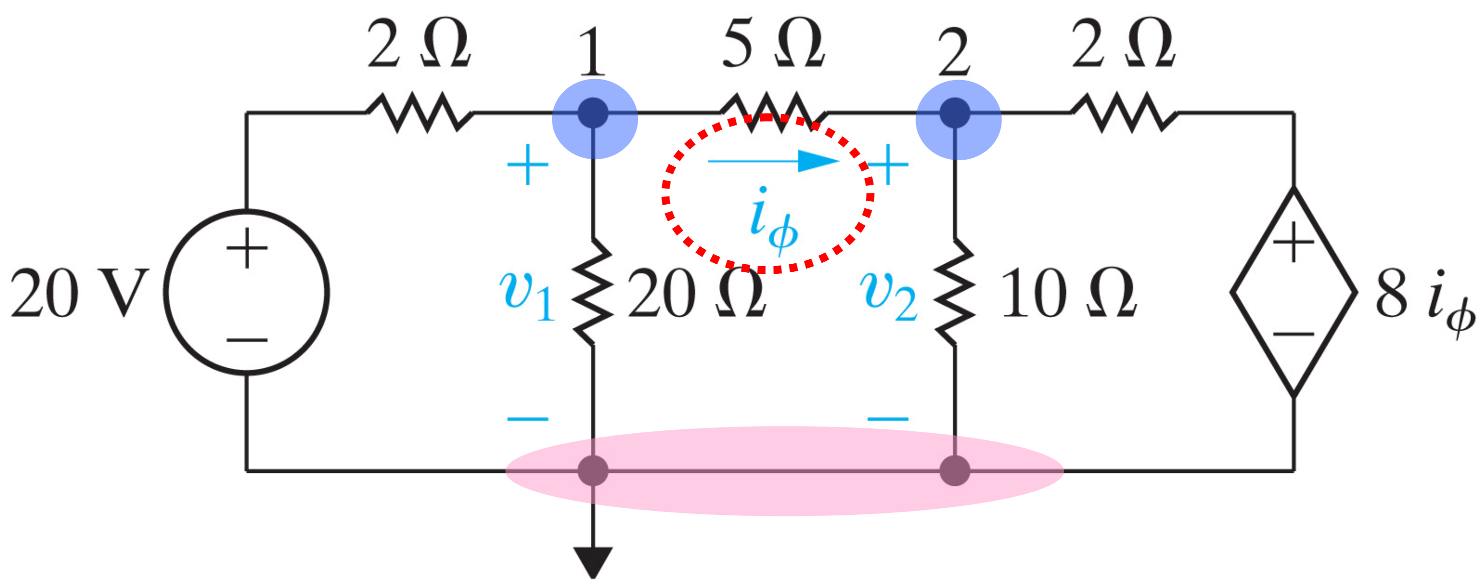
Node 1:
Node 2:

$$\begin{cases} \dfrac{v_1 - 10}{1\,\Omega} + \dfrac{v_1}{5\,\Omega} + \dfrac{v_1 - v_2}{2\,\Omega} = 0, \\ \dfrac{v_2 - v_1}{2\,\Omega} + \dfrac{v_2}{10\,\Omega} = 2. \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1.7 & -0.5 \\ -0.5 & 0.6 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix},$$

$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 9.09 \\ 10.91 \end{bmatrix} \text{V}.$$

NVM in the presence of dependent sources



Node 1:

Node 2:

Constraint:

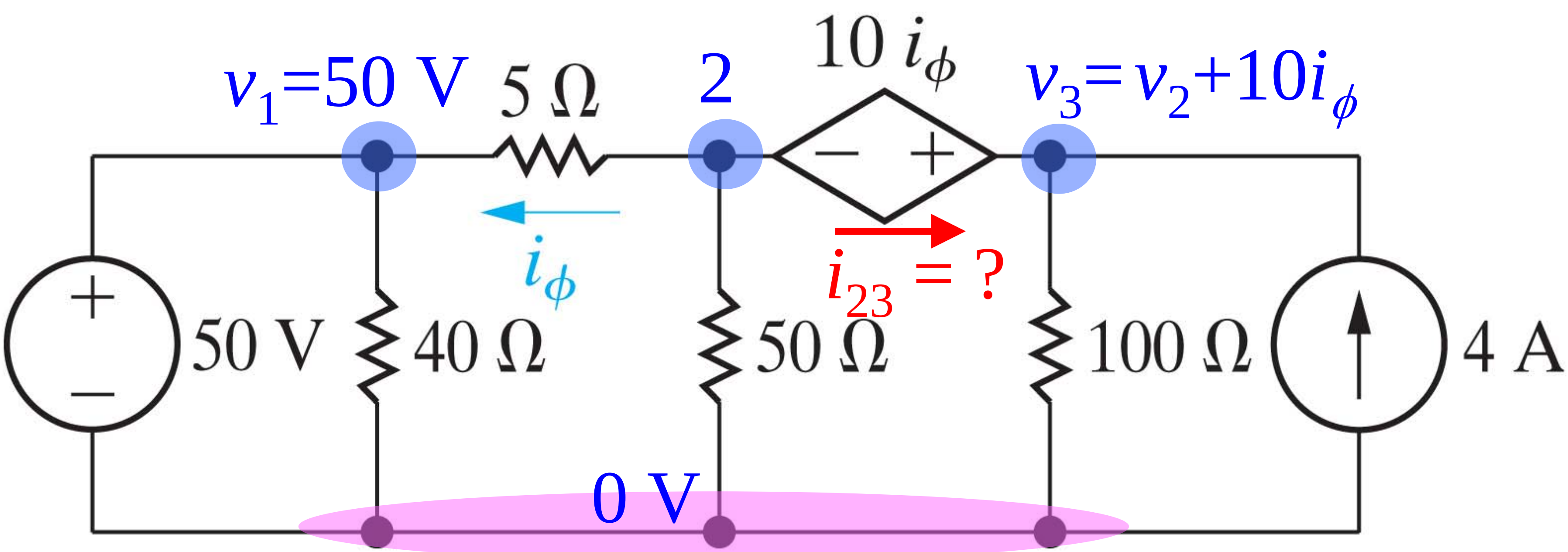
$$\begin{cases} \dfrac{v_1 - 20}{2\,\Omega} + \dfrac{v_1}{20\,\Omega} + \dfrac{v_1 - v_2}{5\,\Omega} = 0, \\[2ex] \dfrac{v_2 - v_1}{5\,\Omega} + \dfrac{v_2}{10\,\Omega} + \dfrac{v_2 - 8i_\phi}{2\,\Omega} = 0, \\[2ex] i_\phi = \dfrac{v_1 - v_2}{5\,\Omega}. \end{cases}$$

$$\Rightarrow \begin{bmatrix} 0.75 & -0.2 \\ -1 & 1.6 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix},$$

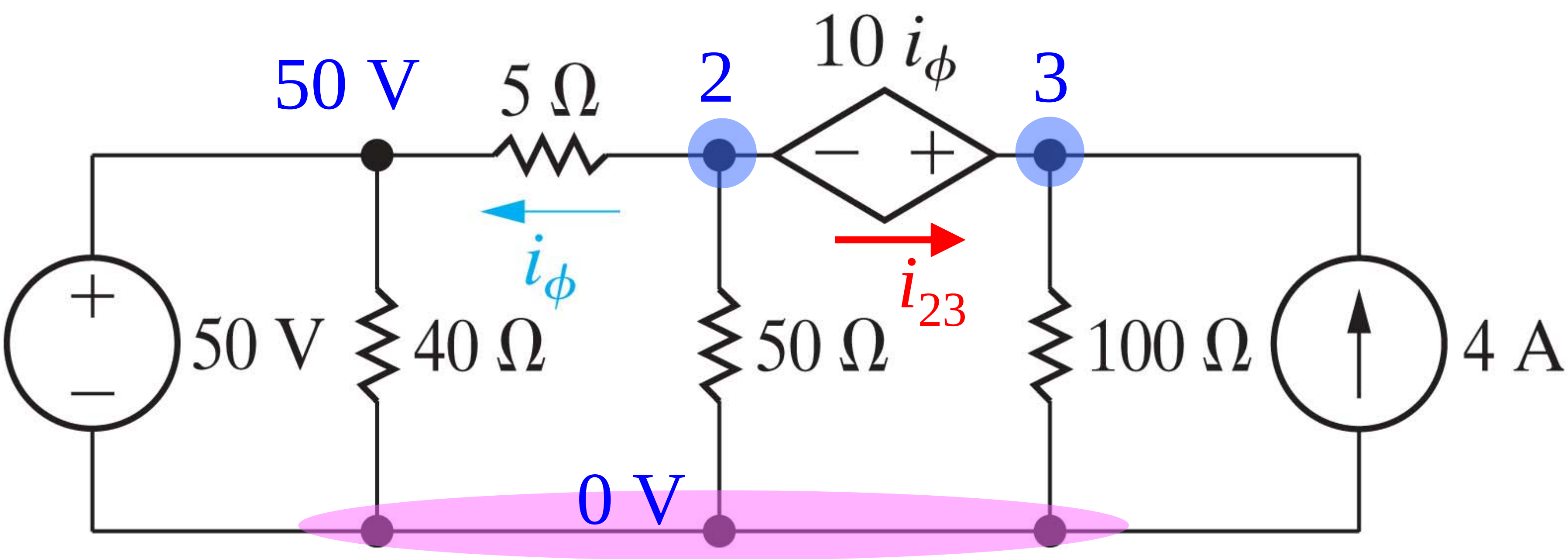
$$\Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \text{V}, \; i_\phi = 1.2 \text{ A}.$$

Case of failing to derive node equation

- When a voltage source (either independent or dependent) is the only element between two essential nodes, the essential branch current is undetermined, $\Rightarrow$ fail to apply KCL to either node!
- E.g. i_{23} is undetermined, fail to apply KCL to Nodes 2, 3.



Solution 1: Add an unknown current



Node 2: $\begin{cases} \dfrac{v_2 - 50}{5\,\Omega} + \dfrac{v_2}{50\,\Omega} + \cancel{i_{23}} = 0, \\ \dfrac{v_3}{100\,\Omega} = \cancel{i_{23}} + 4. \end{cases}$ $\Rightarrow \dfrac{v_2 - 50}{5\,\Omega} + \dfrac{v_2}{50\,\Omega} + \dfrac{v_3}{100\,\Omega} = 4 \cdots (1)$

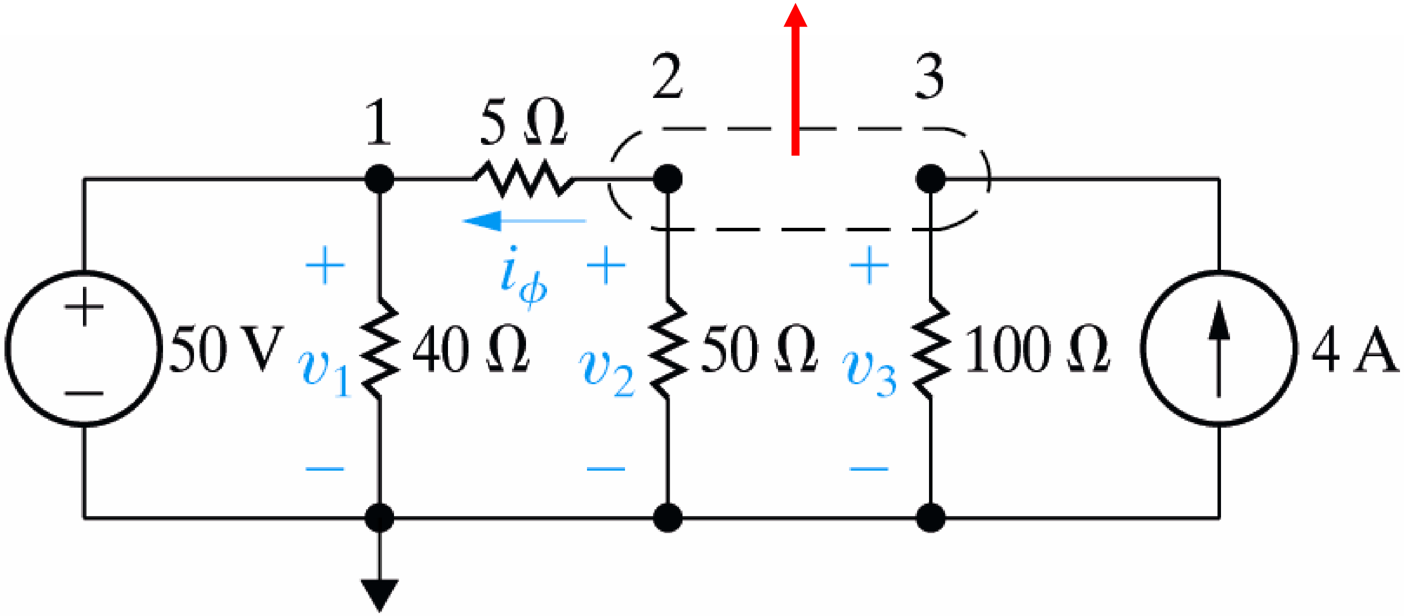
Node 3:

Source constraint: $v_3 = v_2 + 10(v_2 - v_1)/(5\ \Omega)\ldots(2)$.

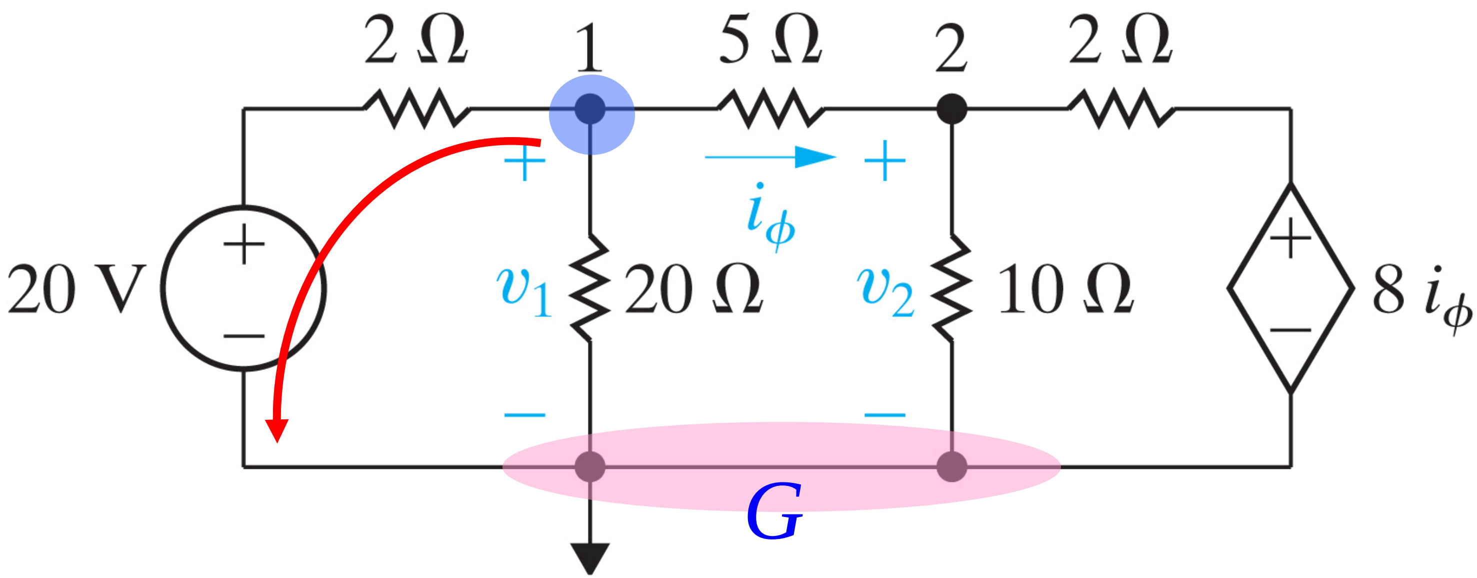
Solution 2: Use supernode

- By applying KCL to a supernode formed by combination of two essential nodes, one can get the same equation without the intermediate step.

$$\frac{v_2-50}{5\,\Omega}+\frac{v_2}{50\,\Omega}+\frac{v_3}{100\,\Omega}=4\cdots(1)$$

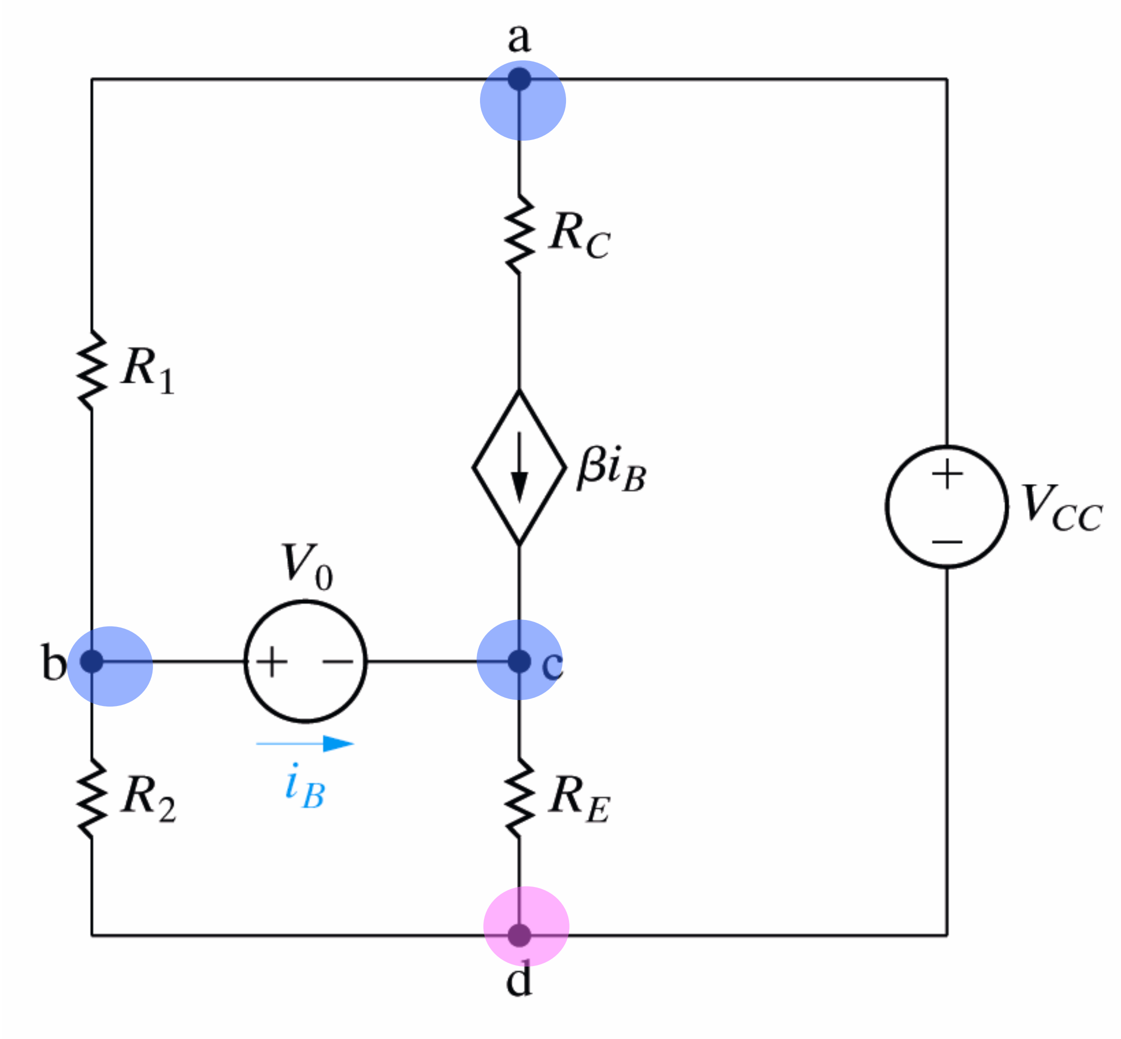


Counter example (Example 4.3)



20-V source is not the only element between Nodes 1 and *G*, ⇒ branch current $i_{20V} = (v_1 - 20)/(2\ \Omega)$ is still available, KCL can still be applied to Node 1, no need to use supernode.

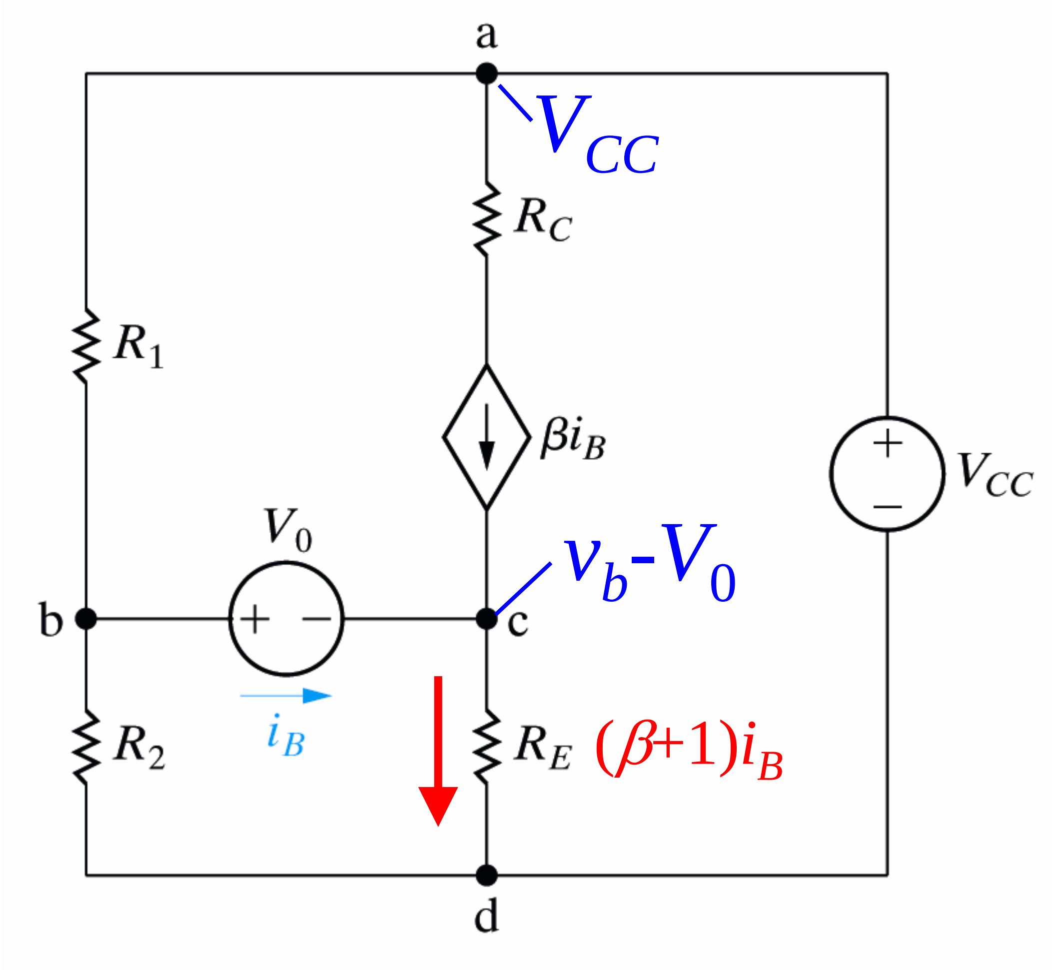
Example 2.11: Amplifier circuit (1)



- n_e=4, $\Rightarrow$ 3 unknown voltages. Since i_B cannot be derived by node voltages, $\Rightarrow$ 4 unknowns.
- The 2 voltage sources provide 2 constraints:

$$\begin{cases} v_a = V_{CC} \cdots (1) \\ v_c = v_b - V_0 \cdots (2) \end{cases}$$

Example 2.11 (2)



- Apply KCL to Node b:

$$\frac{v_b}{R_2}+\frac{v_b-V_{CC}}{R_1}+i_B=0\cdots(3)$$

- Apply KCL to Node c:

$$i_{cd}=(1+\beta)i_B,$$

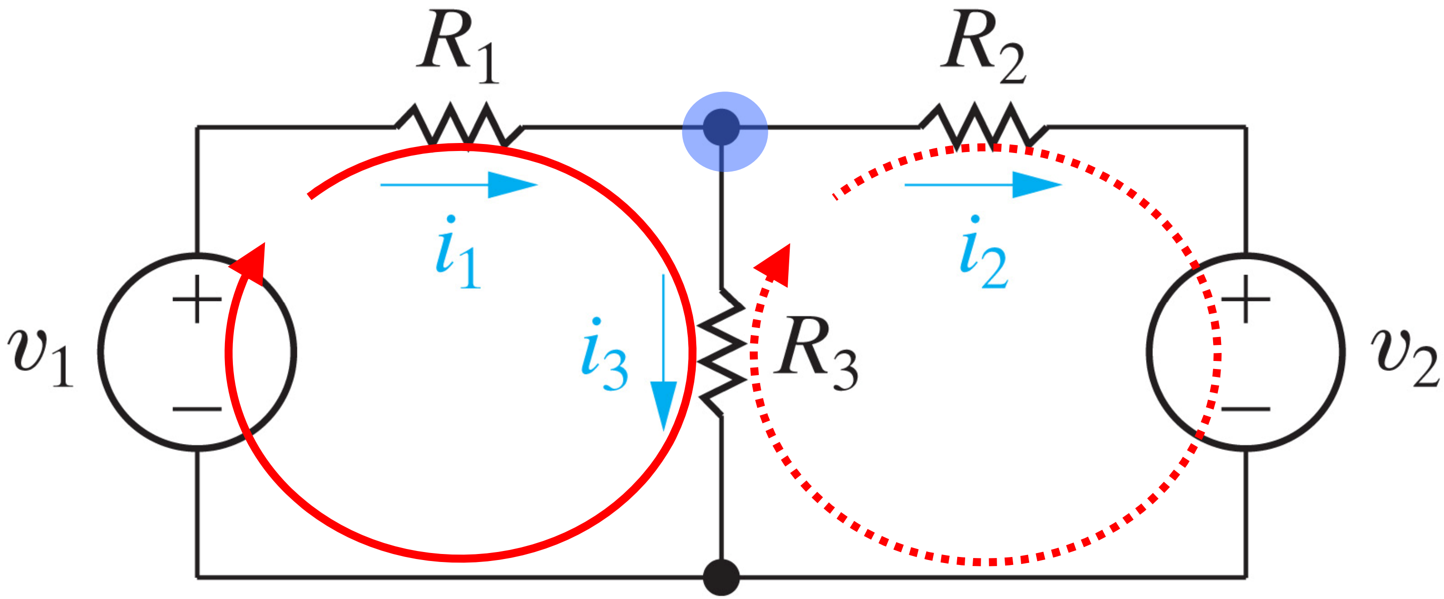
$$\Rightarrow(1+\beta)i_B=\frac{v_c}{R_E}\cdots(4)$$

- Use Supernode bc is also OK.

Section 4.5-4.7
The Mesh-Current Method (MCM)

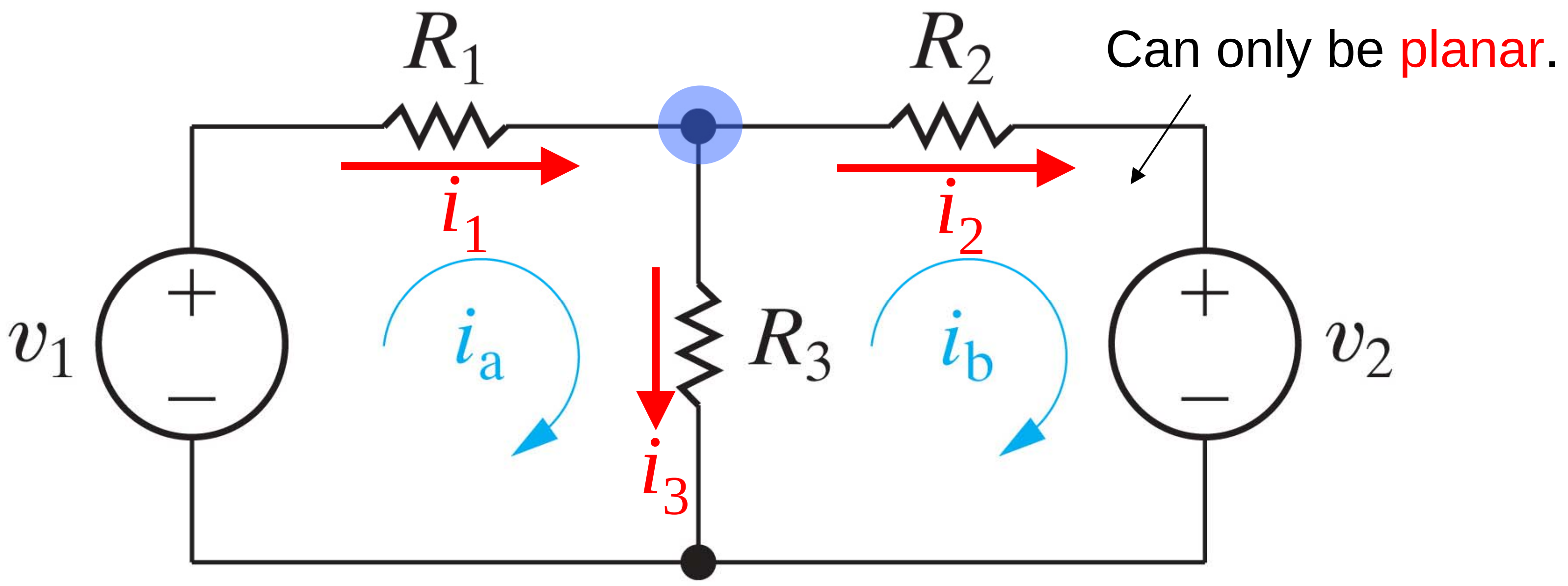
1. Advantage of using mesh currents as unknowns
2. Use of supermesh

Branch currents as unknowns



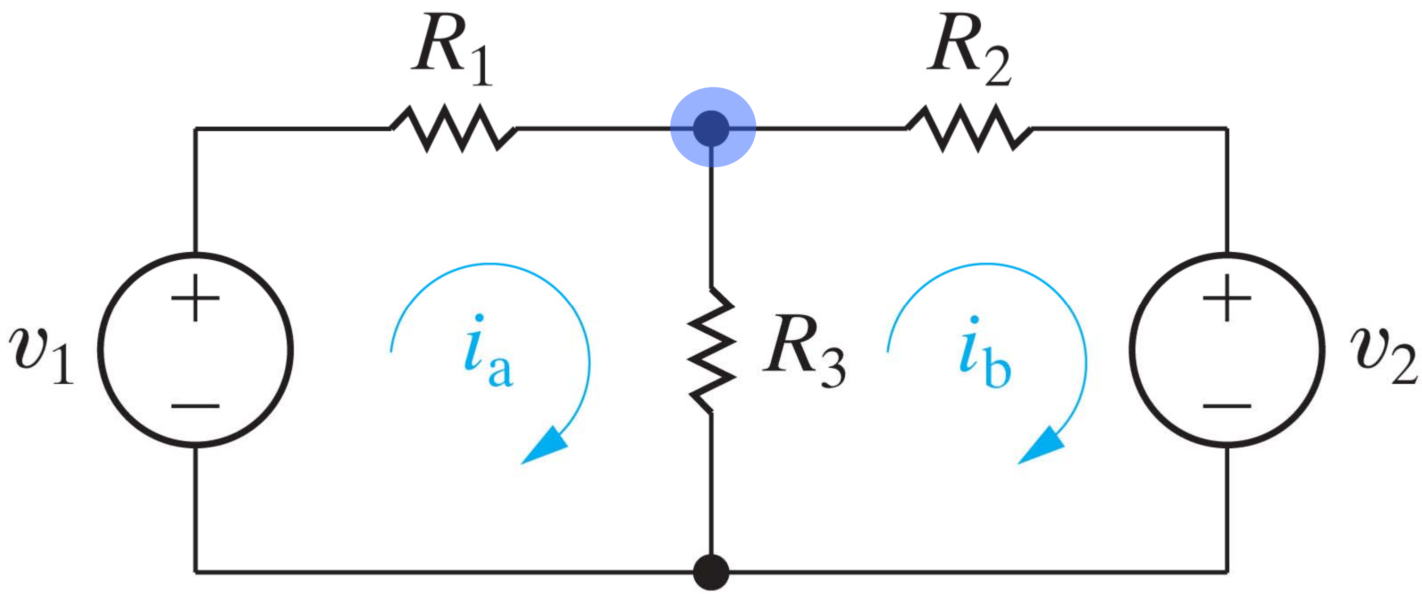
- Number of essential branches $b_e = 3$, number of essential nodes $n_e = 2$.
- To solve $\{i_1,\ i_2,\ i_3\}$, use KCL and KVL to get n_e-1 $= 1$ and b_e-$(n_e$-$1) = 2$ equations.

Advantage of using mesh currents as unknowns



- Relation between branch currents and mesh currents: $i_1 = i_a, \quad i_2 = i_b, \quad i_3 = i_a - i_b.$
- Each mesh current flows into and out of any node on the way, $\Rightarrow$ automatically satisfy KCL.

List b_e-(n_e-1) equations by KVL, Solve them

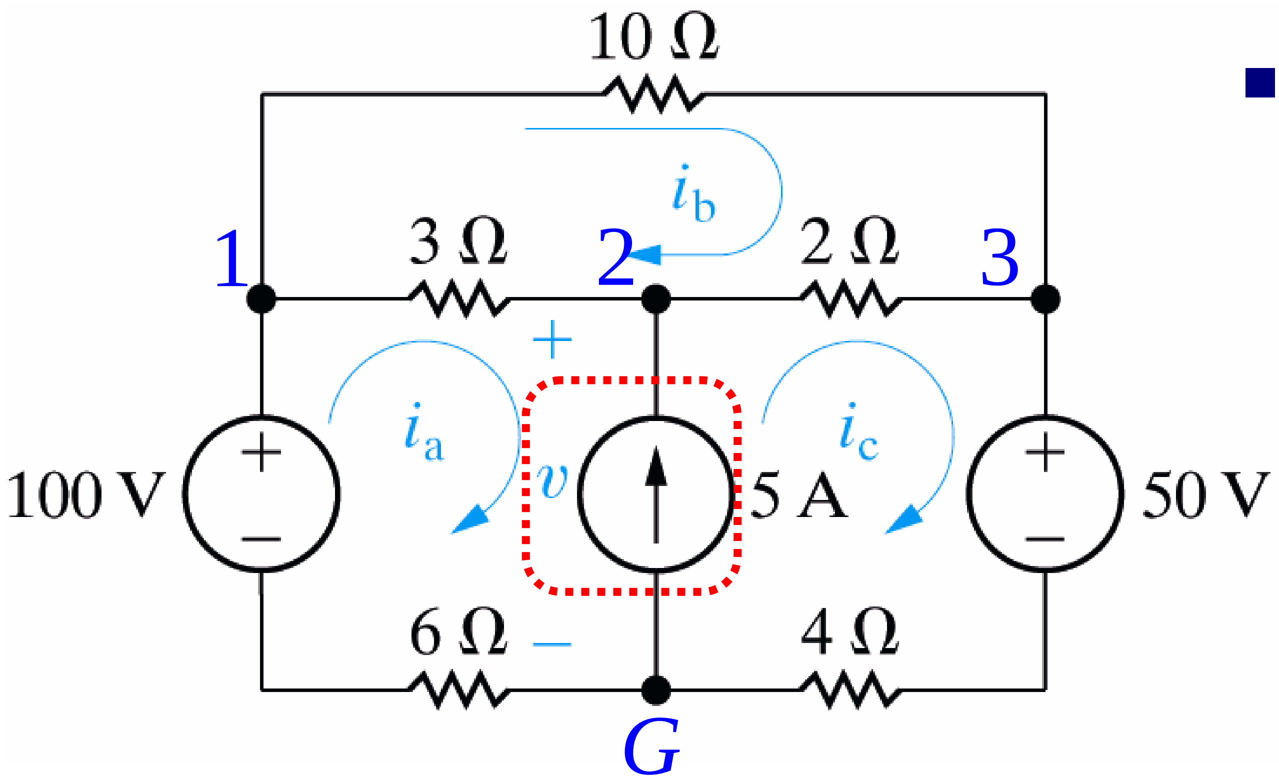


$$\begin{cases} \text{Mesh a: } v_1 = i_a R_1 + (i_a - i_b) R_3 \cdots (1) \\ \text{Mesh b: } i_b R_2 + v_2 + (i_b - i_a) R_3 = 0 \cdots (2) \end{cases}$$

$$\Rightarrow \begin{bmatrix} R_1 + R_3 & -R_3 \\ -R_3 & R_2 + R_3 \end{bmatrix} \times \begin{bmatrix} i_a \\ i_b \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}, \begin{bmatrix} i_a \\ i_b \end{bmatrix} = [R]^{-1} \times \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}.$$

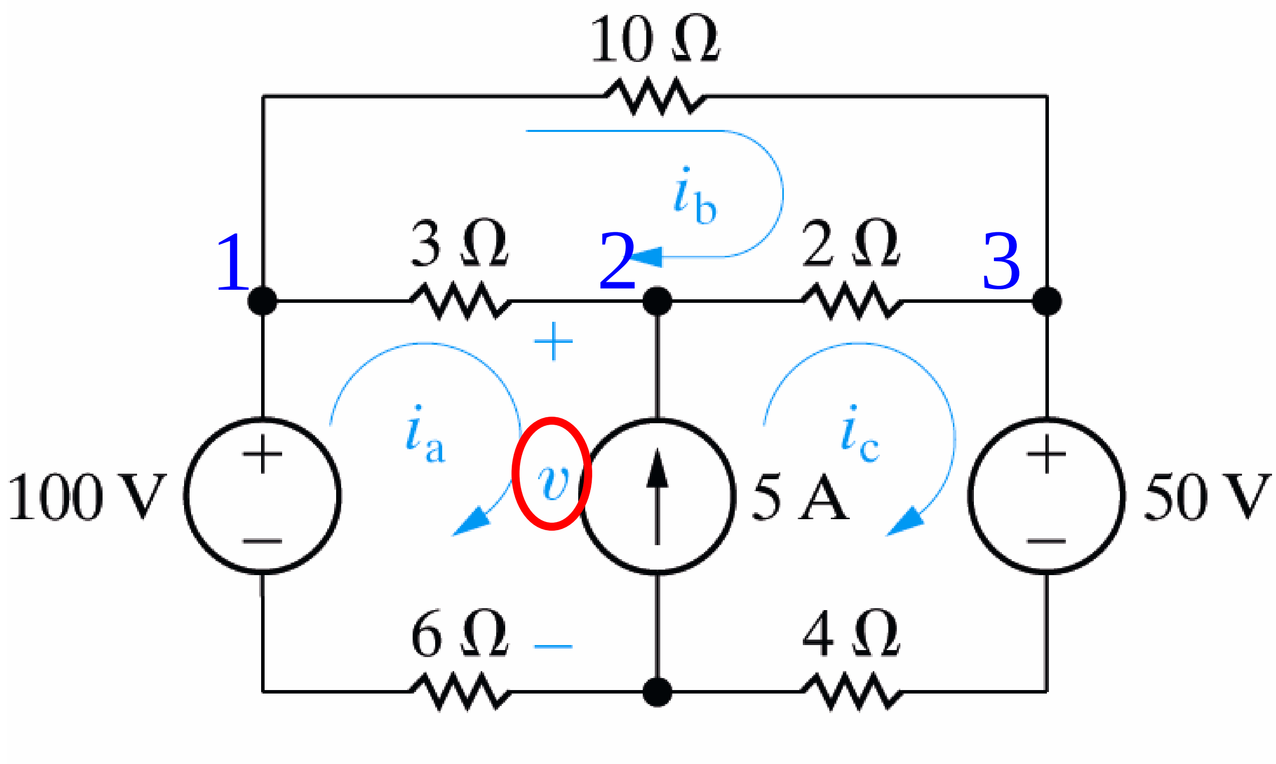
Case of failing to derive mesh equation

- When a current source is between two essential nodes (no need to be the only element), the voltage drop across the source is undetermined, ⇒ fail to apply KVL to either mesh!



- E.g. v_{2G} is undetermined, fail to apply KVL to Meshes a and c.

Solution 1: Add an unknown voltage



$$\begin{aligned}&\text{Mesh a:}\quad \left\{\begin{array}{l}(3\,\Omega)(i_a - i_b) + \cancel{v} + (6\,\Omega)i_a = 100,\\[4pt] (2\,\Omega)(i_c - i_b) + 50 + (4\,\Omega)i_c = \cancel{v}.\end{array}\right.\\ &\text{Mesh c:}\end{aligned}$$

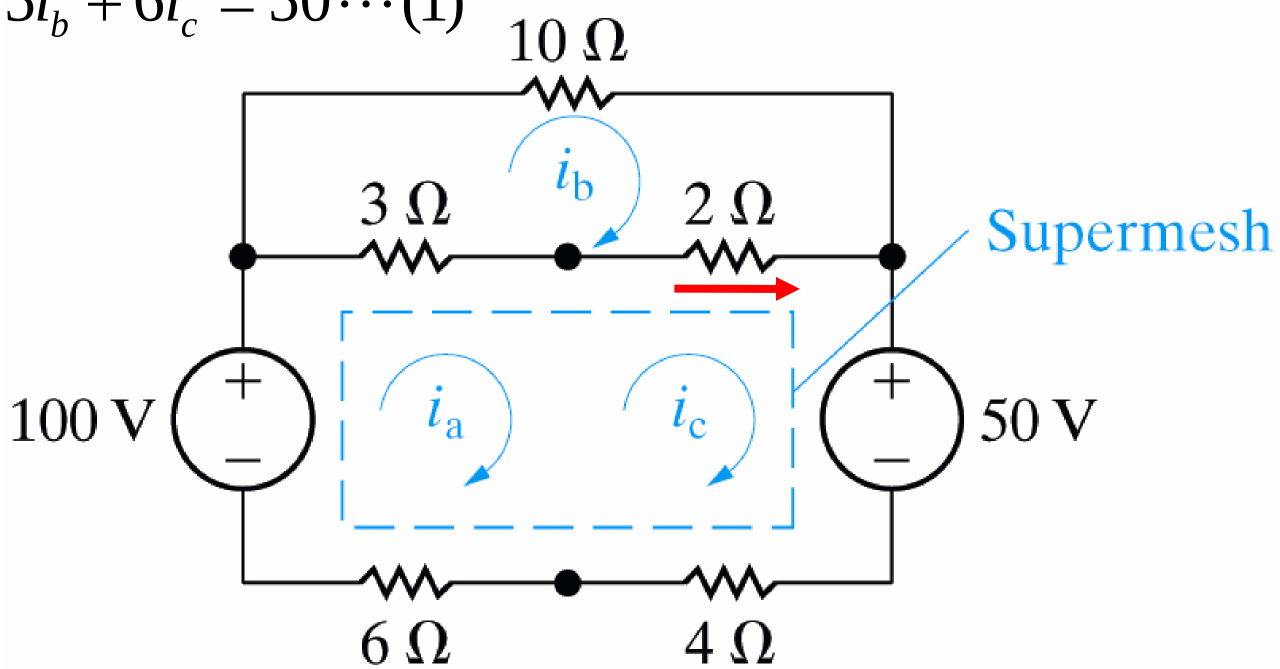
$$\Rightarrow 9i_a - 5i_b + 6i_c = 50 \cdots (1)$$

Solution 2: Use supermesh

- By applying KVL to a supermesh formed by combination of two meshes, one can get the same equation without the intermediate step.

$$(3\,\Omega)(i_a - i_b) + (2\,\Omega)(\underline{i_c - i_b}) + 50 + (4\,\Omega)i_c + (6\,\Omega)i_a = 100,$$

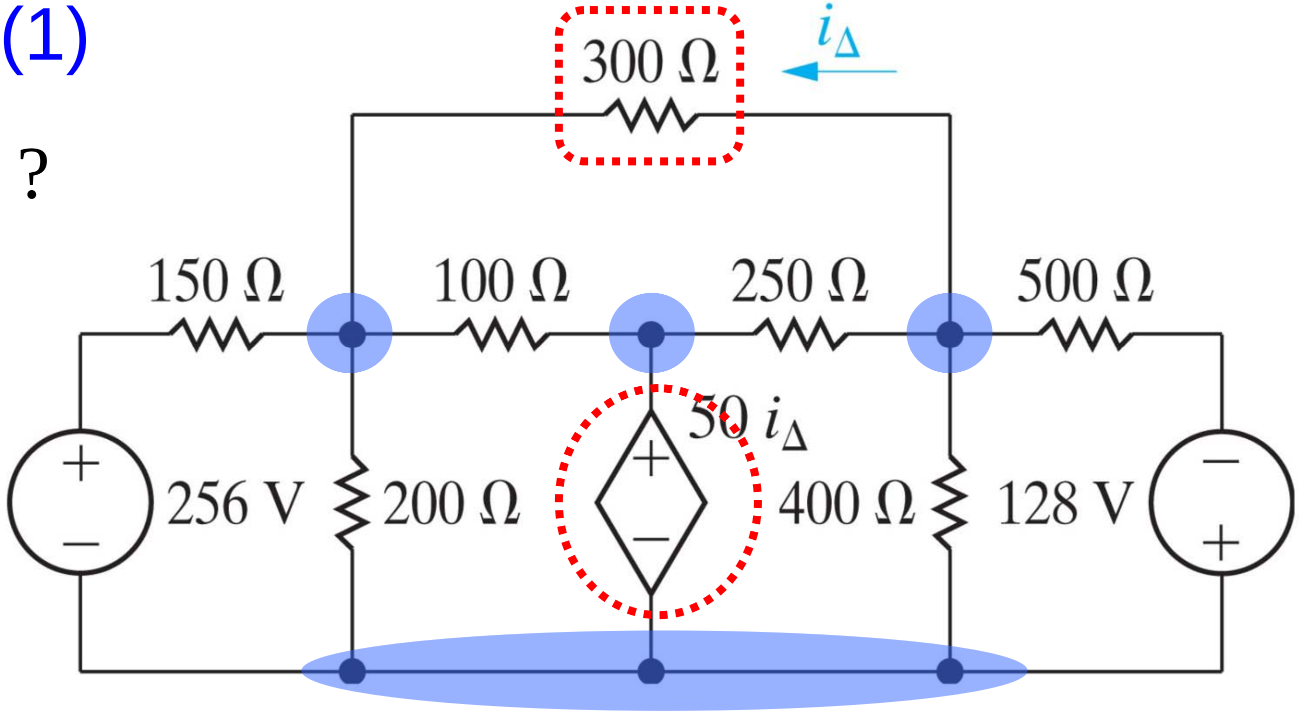
$$\Rightarrow 9i_a - 5i_b + 6i_c = 50 \cdots (1)$$



Section 4.8
The Node-Voltage Method vs. the Mesh-Current Method

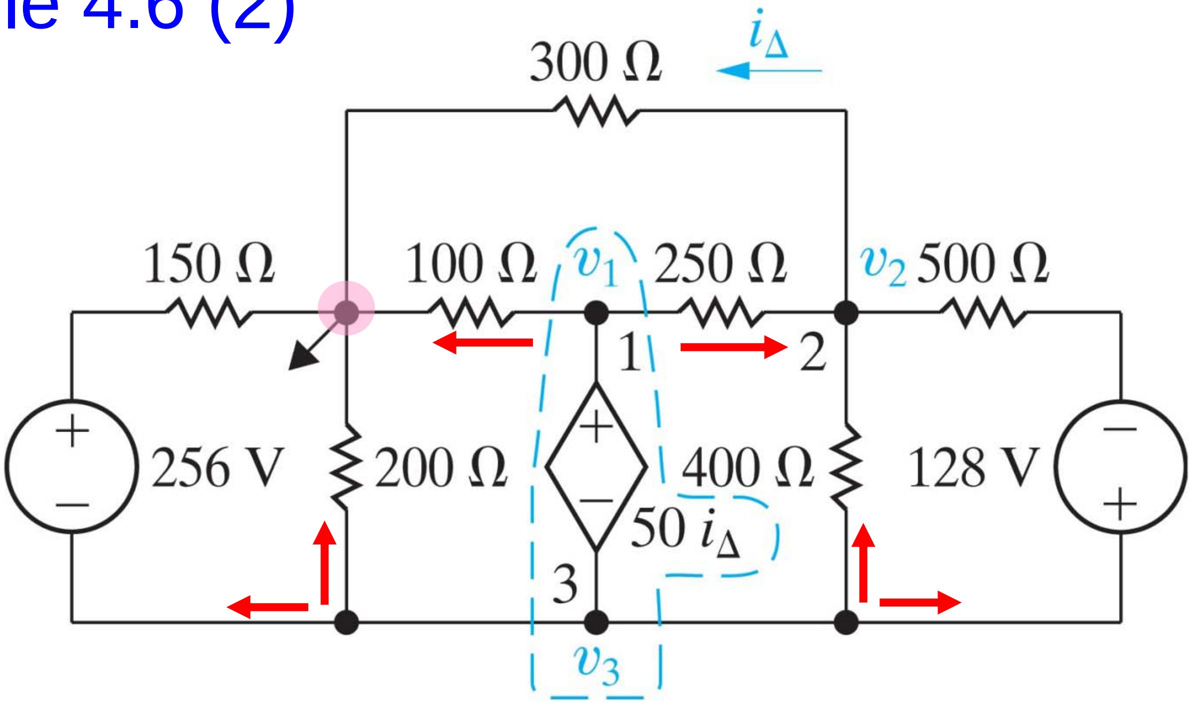
Example 4.6 (1)

- Q: $P_{300\Omega} = ?$



- 5 meshes, no current source (no supermesh). ⇒ MCM needs 5 mesh equations.
- 4 essential nodes, the dependent voltage source is the only element on that branch (1 supernode). ⇒ NVM needs (4-1)-1= 2 node equations.

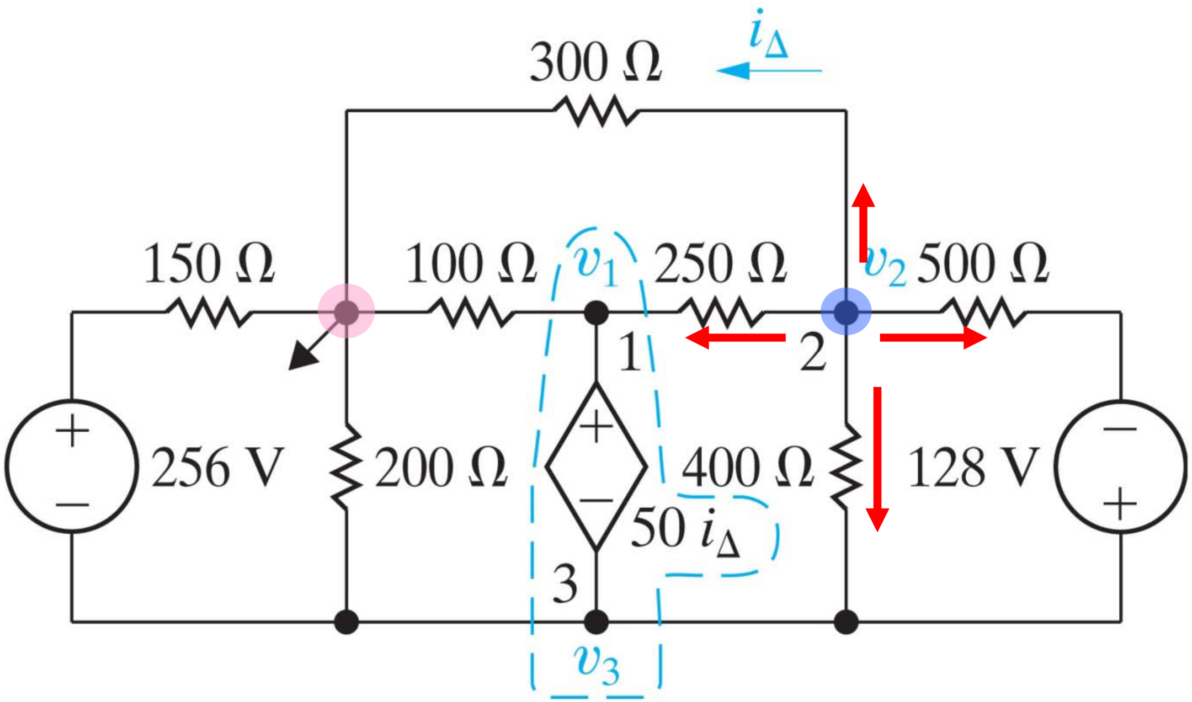
Example 4.6 (2)



- Choose the reference node such that $P_{300\Omega}$ can be calculated by only solving v_2.
- Apply KCL to Supernode 1,3:

$$\frac{v_1}{100\,\Omega}+\frac{v_1-v_2}{250\,\Omega}+\frac{v_3}{200\,\Omega}+\frac{v_3+256}{150\,\Omega}+\frac{v_3-v_2}{400\,\Omega}+\frac{v_3-128-v_2}{500\,\Omega}=0\cdots(1)$$

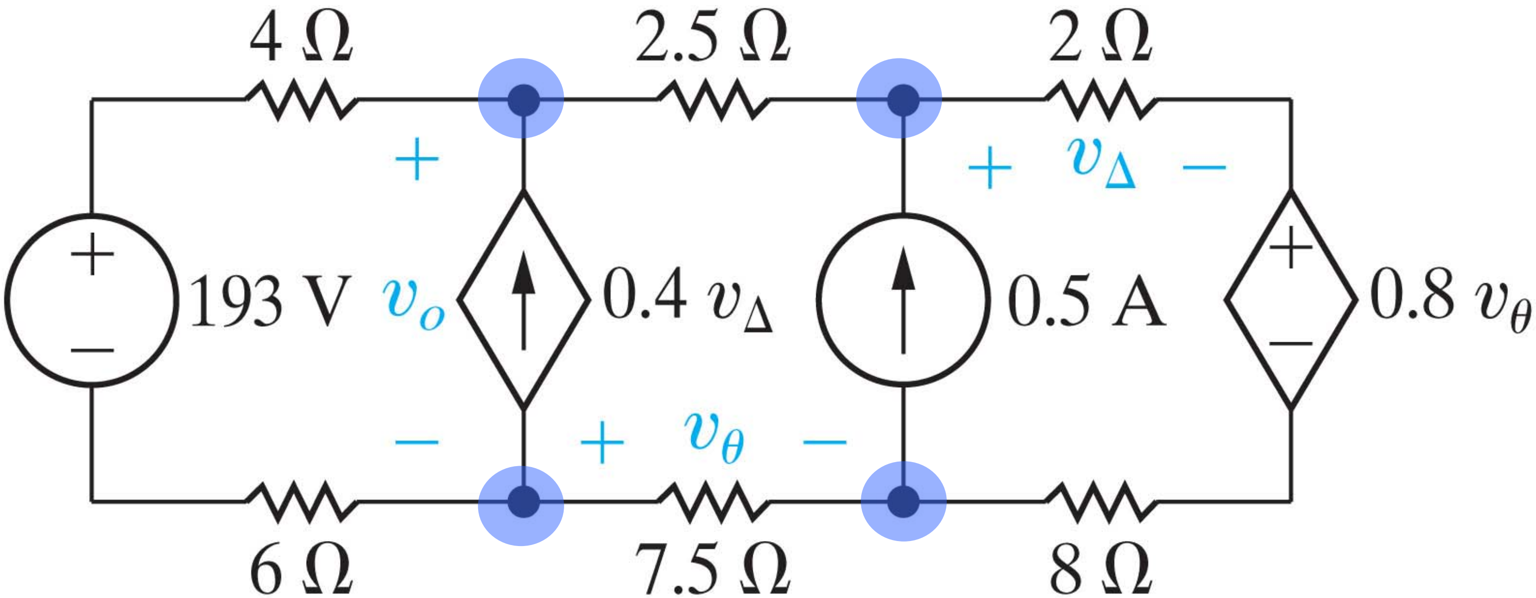
Example 4.6 (3)



Node 2: $\dfrac{v_2}{300\,\Omega}+\dfrac{v_2-v_1}{250\,\Omega}+\dfrac{v_2-v_3}{450\,\Omega}+\dfrac{v_2+128-v_3}{500\,\Omega}=0\cdots(2)$

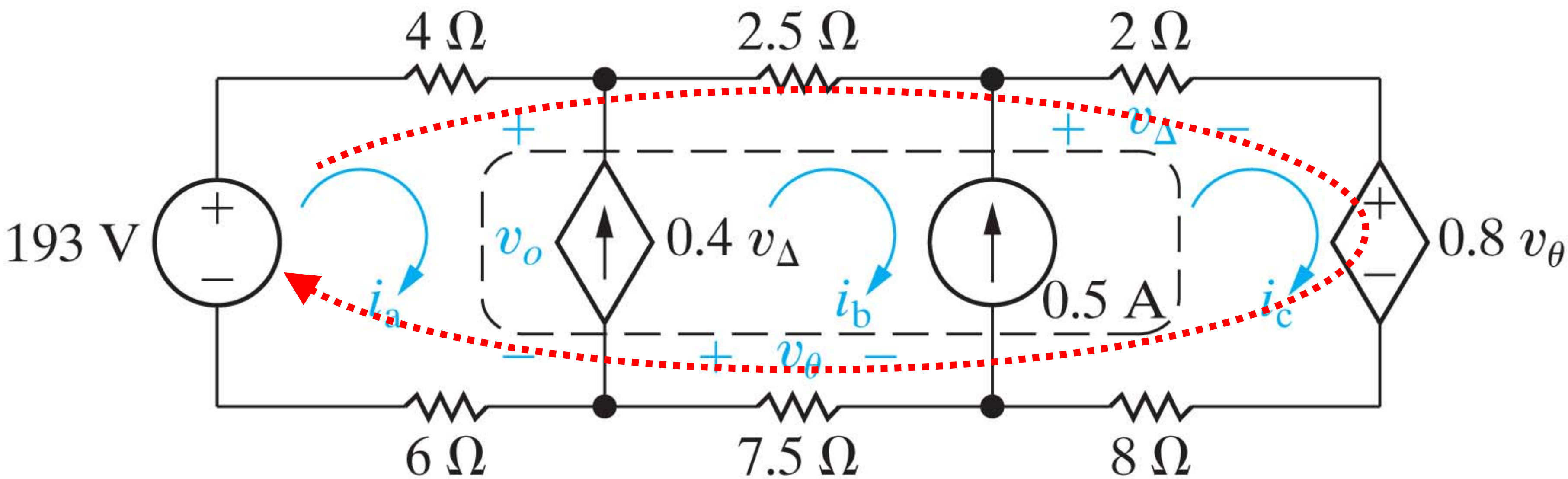
Source constraint: $v_1-v_3=50\dfrac{v_2}{300\,\Omega}=\dfrac{v_2}{6}\cdots(3)$

Example 4.7 (1)



- 3 meshes, 2 current sources (2 supermeshes). $\Rightarrow$ MCM needs 1 mesh equation.
- 4 essential nodes, no voltage source is the only element on one branch (no supernode). $\Rightarrow$ NVM needs (4-1)= 3 node equations.

Example 4.7 (2)



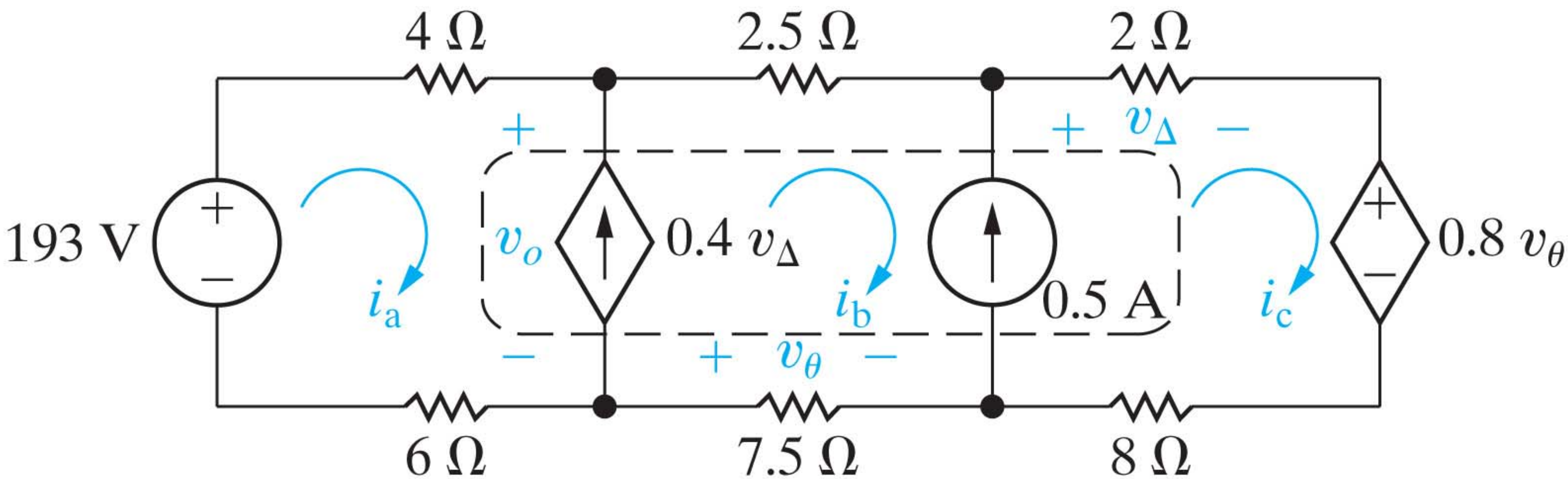
- Apply KVL to Supermesh a, b, c:

$$(4\,\Omega + 6\,\Omega)i_a + (2.5\,\Omega + 7.5\,\Omega)i_b + (2\,\Omega + 8\,\Omega)i_c$$

$$+0.8v_\theta = 193;$$

$$\because v_\theta = -i_b(7.5\,\Omega), \Rightarrow 10i_a + 4i_b + 10i_c = 193 \cdots (1)$$

Example 4.7 (3)



- By the two current source constraints:

$$i_b - i_a = 0.4v_\Delta = 0.4(i_c \times 2\Omega) = 0.8i_c \cdots (2)$$

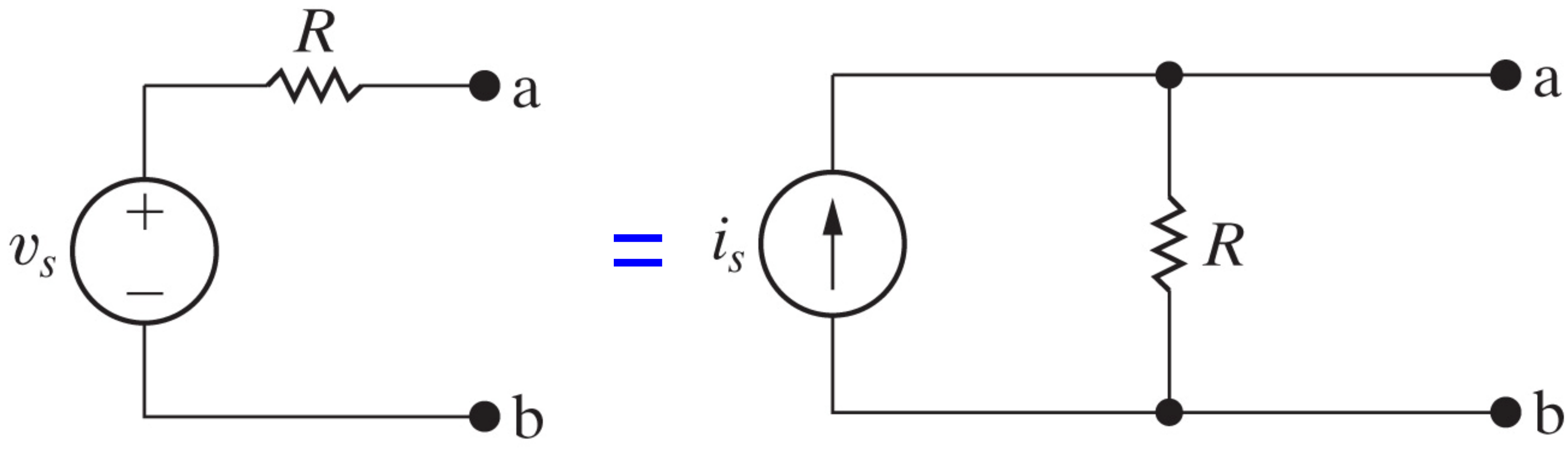
$$i_c - i_b = 0.5 \cdots (3)$$

Section 4.9
Source Transformations

Source transformations

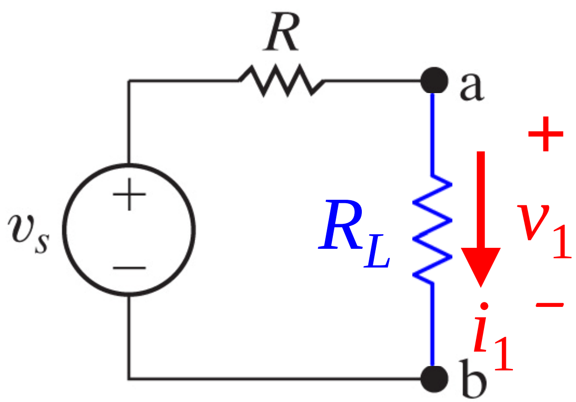
- A voltage source v_s in series with a resistor R can be replaced by a current source i_s in parallel with the same resistor R or vice versa, where

$$i_s = \frac{v_s}{R}$$

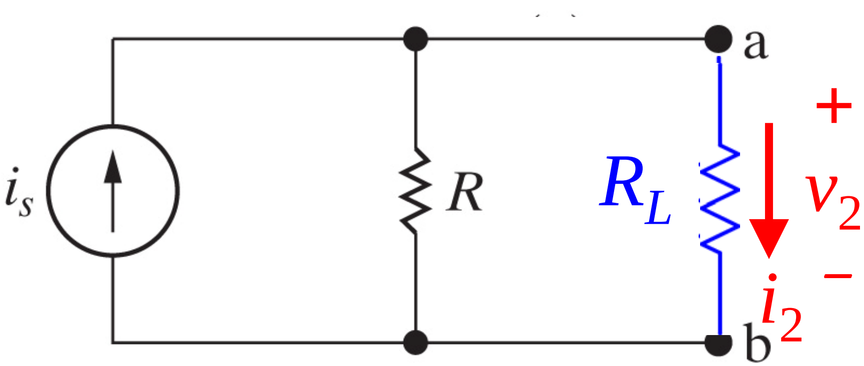


Proof of source transformation

- For any load resistor R_L, the current i and voltage v between terminals a, b should be consistent in both configurations.

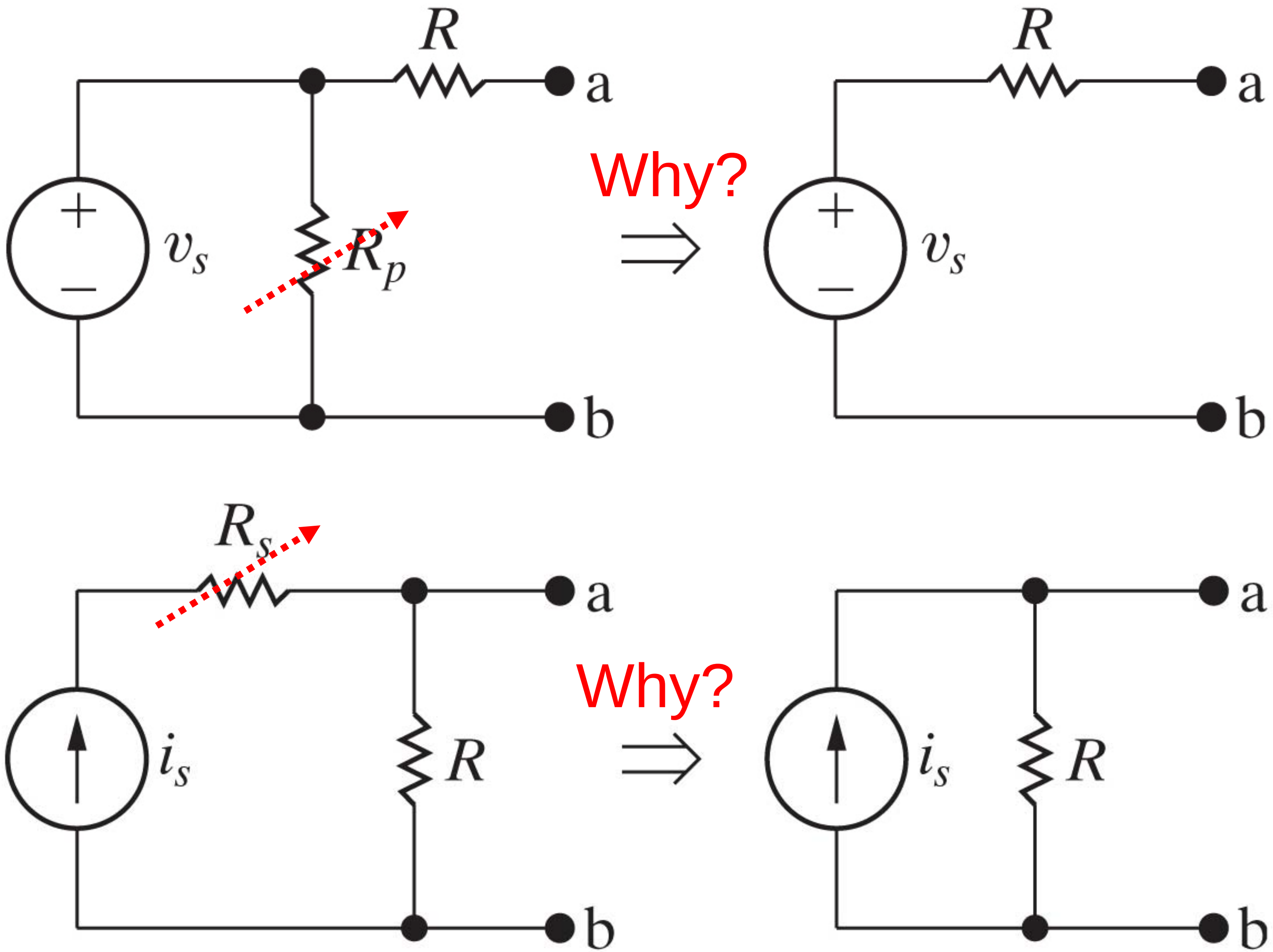


$$i_1 = \frac{v_s}{R+R_L}, \quad v_1 = \frac{R_L}{R+R_L} v_s.$$



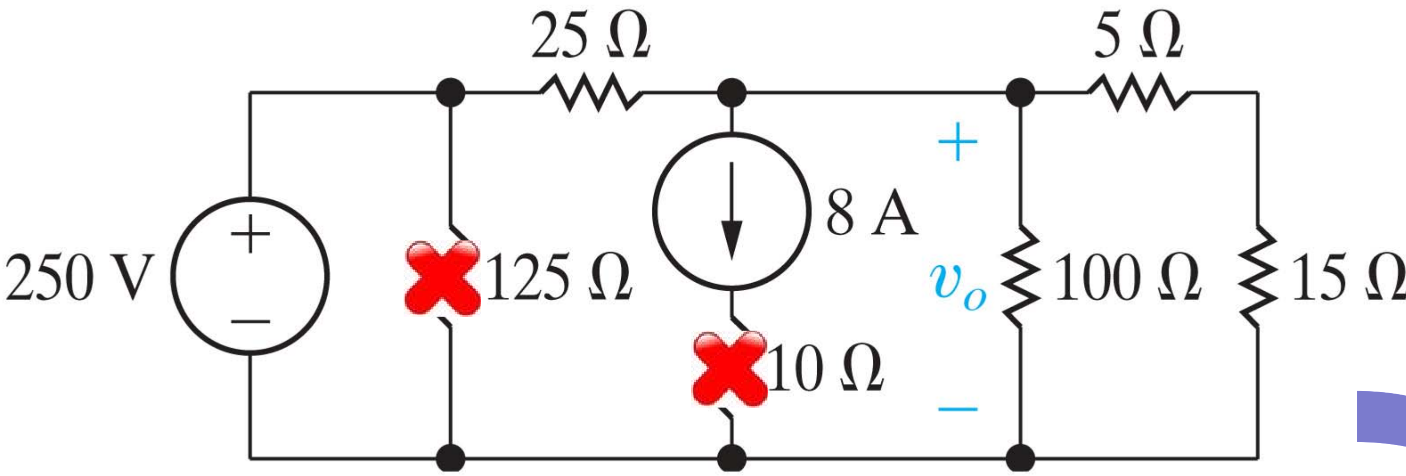
$$\begin{cases} i_2 = \dfrac{R}{R+R_L} i_s = \dfrac{R}{R+R_L} \dfrac{v_s}{R} = i_1, \\ v_2 = i_s (R // R_L) = \dfrac{v_s}{R} \dfrac{RR_L}{R+R_L} = v_1. \end{cases}$$

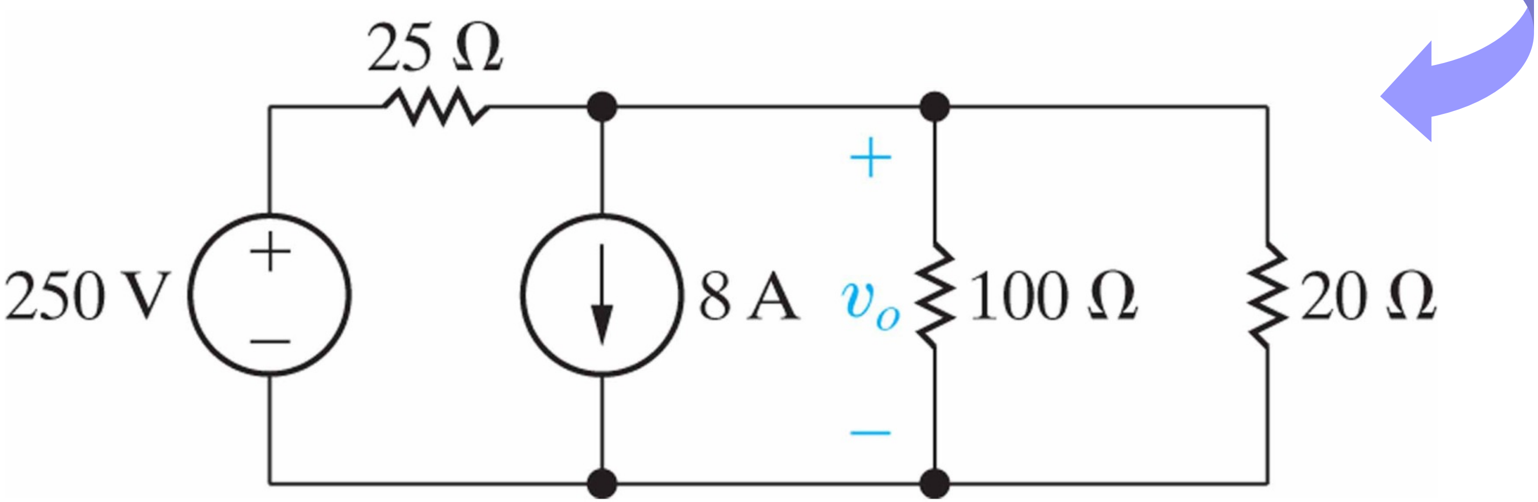
Redundant resistors



Example 4.9 (1)

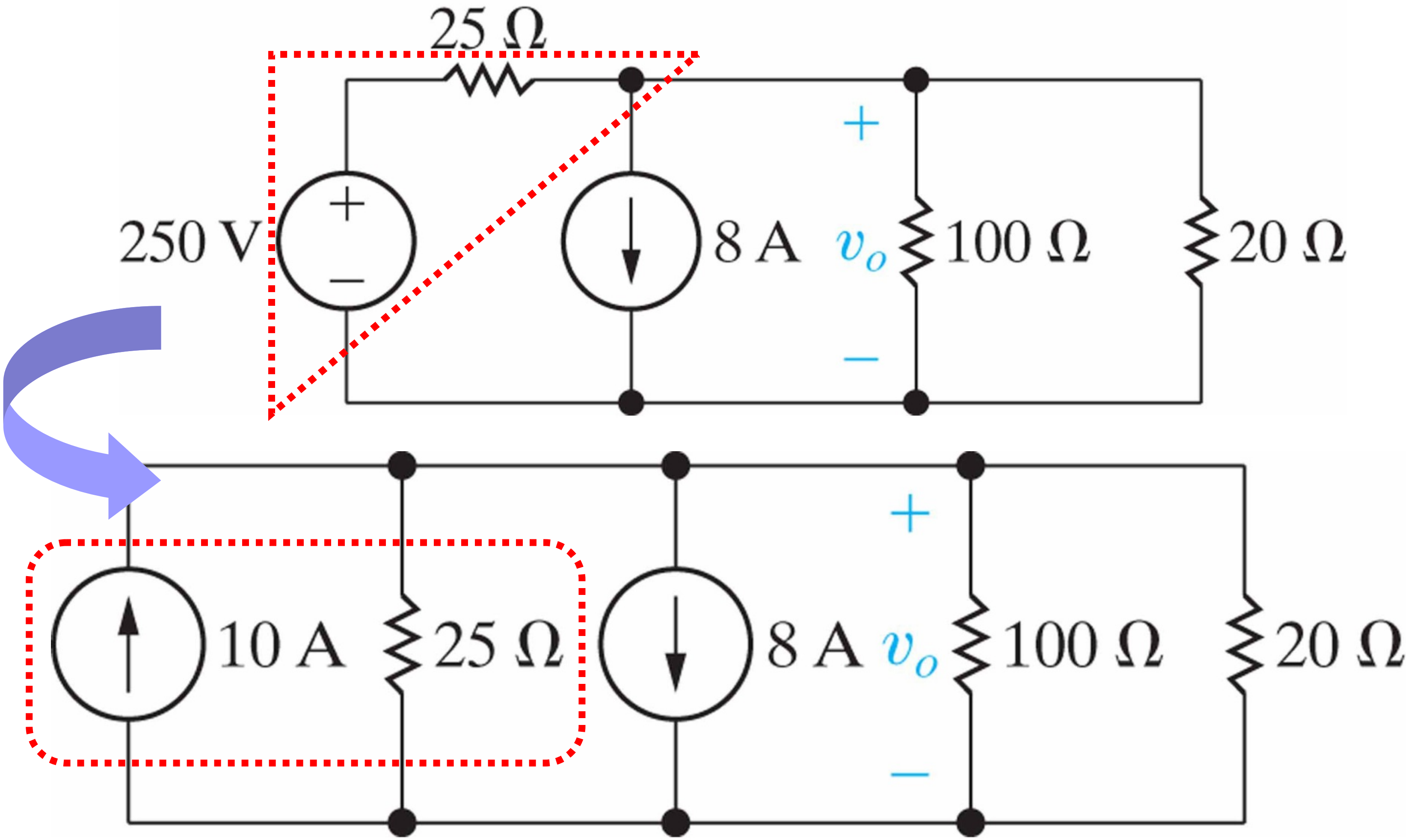
- Q: (1) $v_o = ?$ (2) $P_{250\text{V}} = ?$





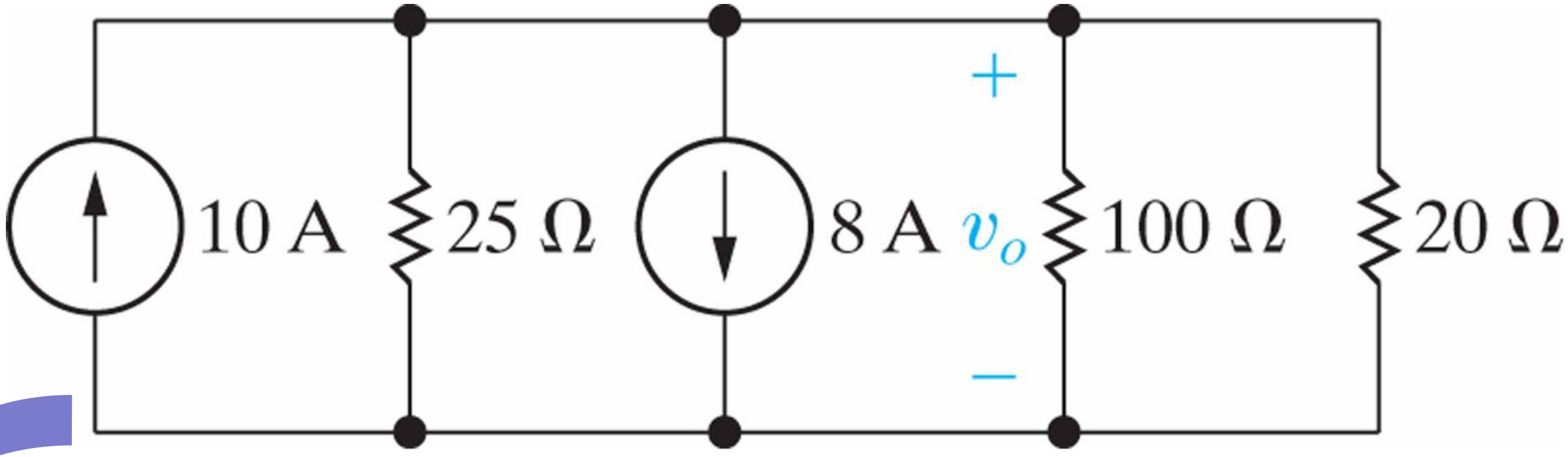
Example 4.9 (2)

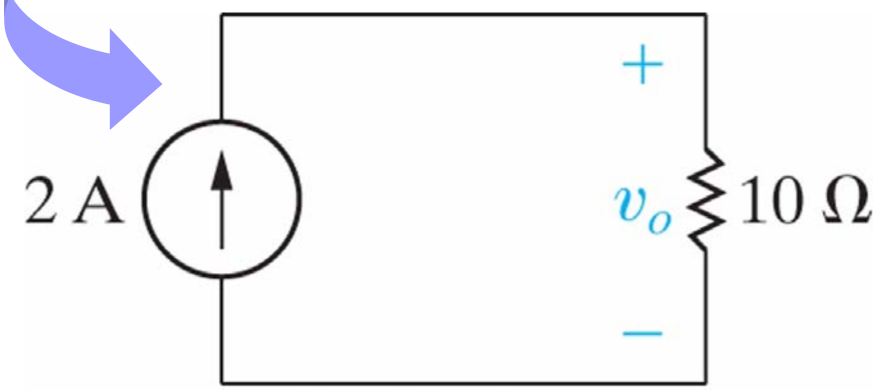
- To find v_o, transform the 250-V voltage source into a 10-A current source.



Example 4.9 (3)

- Now v_o is simply the voltage of the total load.

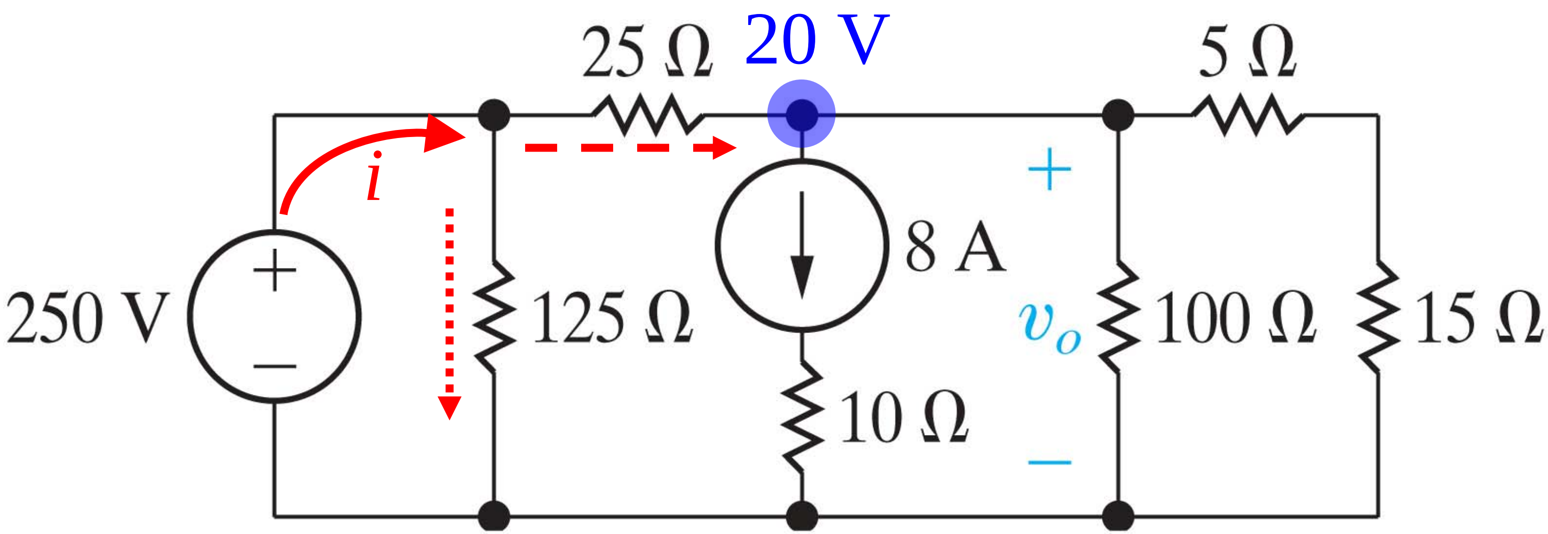




$$v_o = (2\ \text{A}) \times (10\ \Omega) = 20\ \text{V}.$$

Example 4.9 (4)

- $P_{250\text{V}}$ has to be calculated by the original circuit:



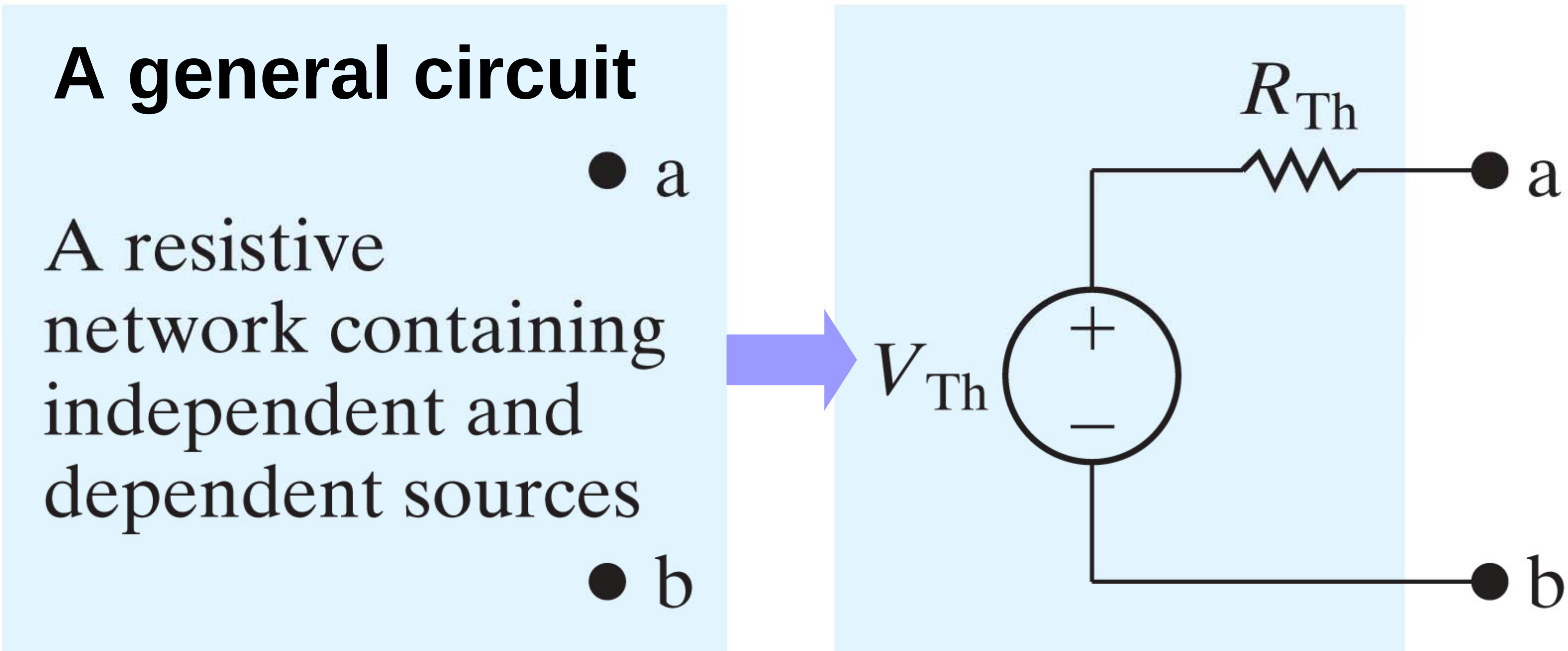
$$i = \frac{250\text{ V}}{125\,\Omega} + \frac{(250-20)\text{ V}}{25\,\Omega} = 11.2\text{ A},$$

$$P_{250\text{V}} = -(11.2\text{ A}) \times (250\text{ V}) = -2.8\text{ kW}.$$ (Power extraction)

Section 4.10, 4.11 Thévenin and Norton Equivalent

1. Definition of equivalent circuit
2. Methods to get the two Thévenin equivalent circuit parameters v_{Th}, R_{Th}
3. Methods to get Thévenin resistance R_{Th} alone
4. Example of applications

Thévenin equivalent circuit

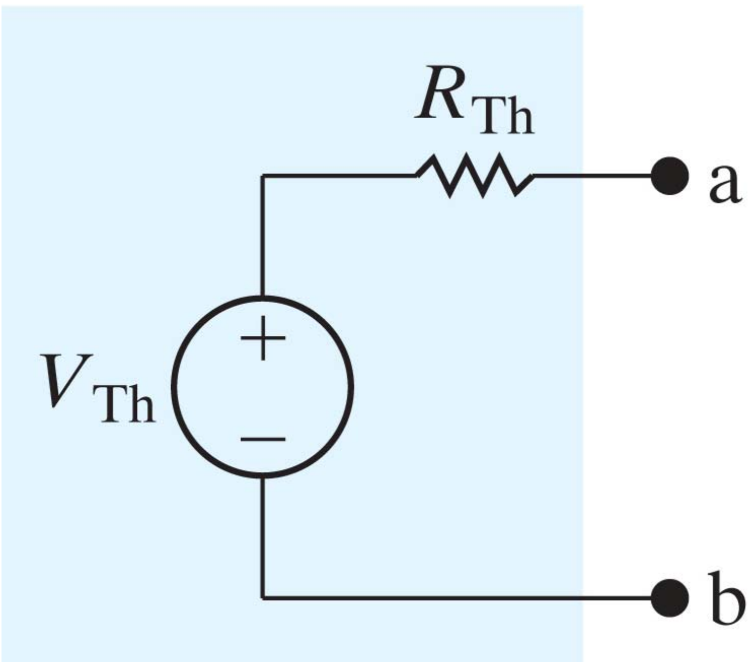


- For any load resistor R_L, the current i and voltage v between terminals a, b should be consistent in both configurations.

Method 1 to get V_{Th} and R_{Th}

- Find open circuit voltage $\Rightarrow v_{oc} = V_{Th}$
- Find short circuit current $\Rightarrow i_{sc}$

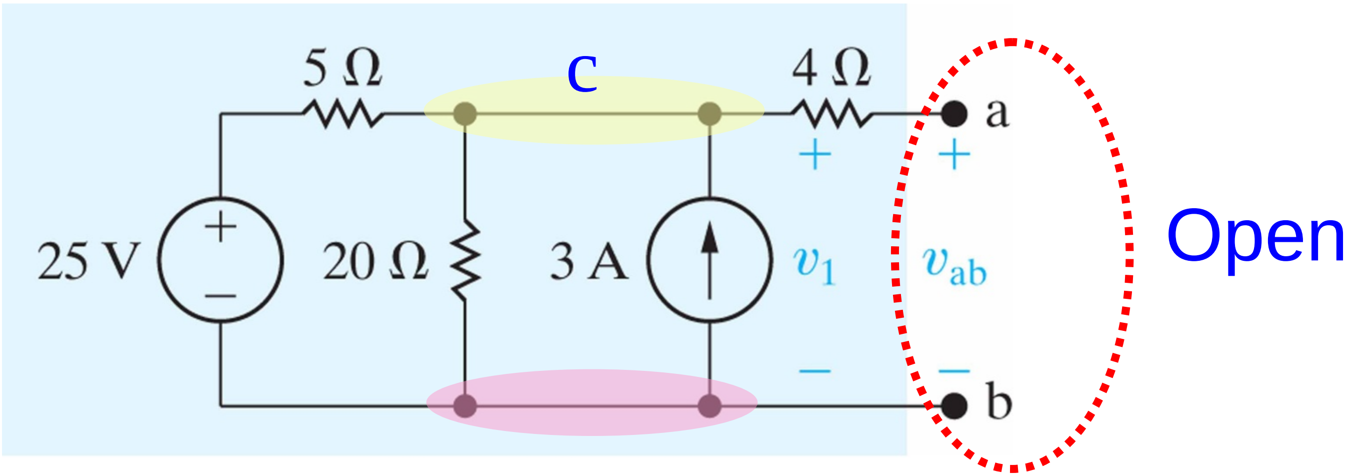
$$\left.\begin{array}{l} v_{oc} = V_{Th} \\ i_{sc} \end{array}\right\} \Rightarrow R_{Th} = \frac{v_{oc}}{i_{sc}}$$



Note: This method is invalid when the circuits contain only dependent sources.

Example: Calculating v_{oc}

- Let terminals a, b be open, $\Rightarrow$ no current flows through the 4-Ω resistor, $\Rightarrow v_{oc} = v_1$.

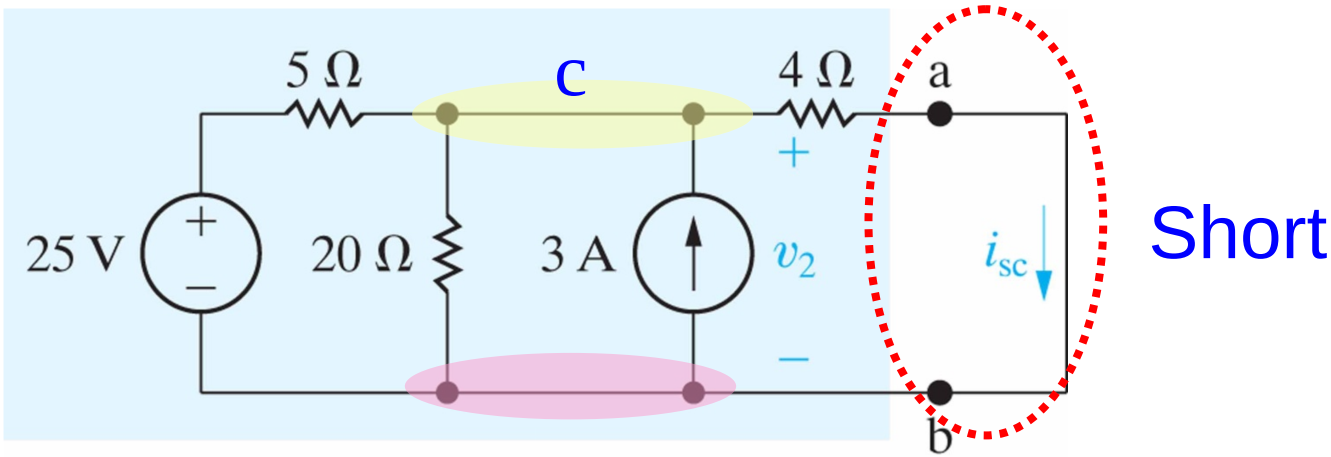


- By NVM, the only unknown is the node voltage v_1. Apply KCL to Node c:

$$\frac{v_1 - 25}{5\,\Omega} + \frac{v_1}{20\,\Omega} = 3, \Rightarrow v_1 = 32\text{ V} = V_{Th}.$$

Example: Calculating i_{sc}

- Let terminals a, b be short, $\Rightarrow i_{sc} = v_2/(4\ \Omega)$.



- By NVM, the only unknown is node voltage v_2. Apply KCL to Node c:

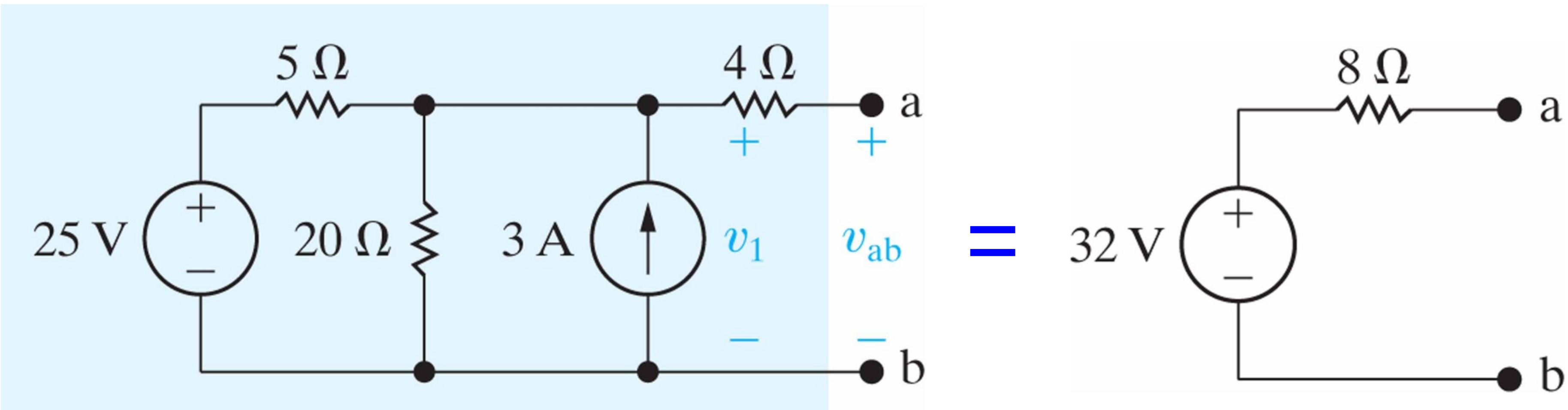
$$\frac{v_2 - 25}{5\,\Omega} + \frac{v_2}{20\,\Omega} + \frac{v_2}{4\,\Omega} = 3, \ \Rightarrow v_2 = 16\text{ V},$$

$$i_{sc} = v_2 / (4\,\Omega) = 4\text{ A}.$$

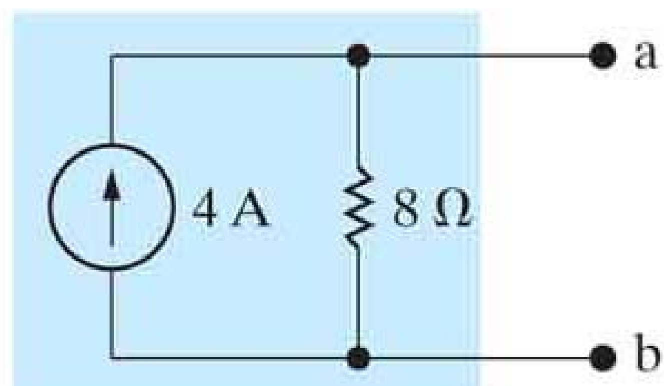
Example: Calculating R_{Th} and Norton circuit

- The Thévenin resistance is:

$$R_{Th} = V_{Th} / i_{sc} = (32\text{ V})/(4\text{ A}) = 8\ \Omega.$$

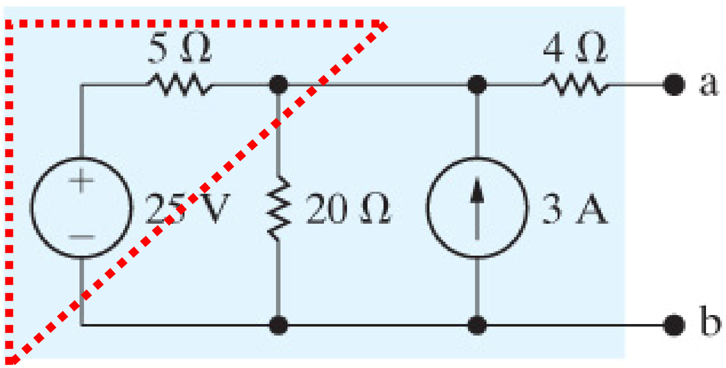


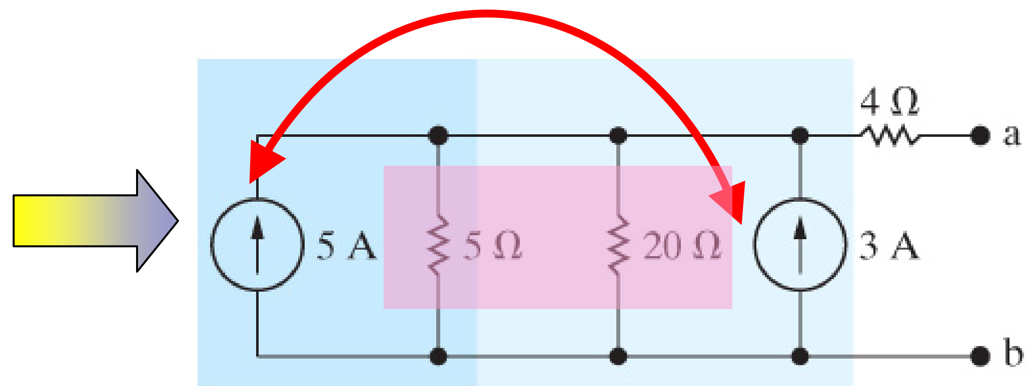
- The Norton equivalent circuit can be derived by source transformation:

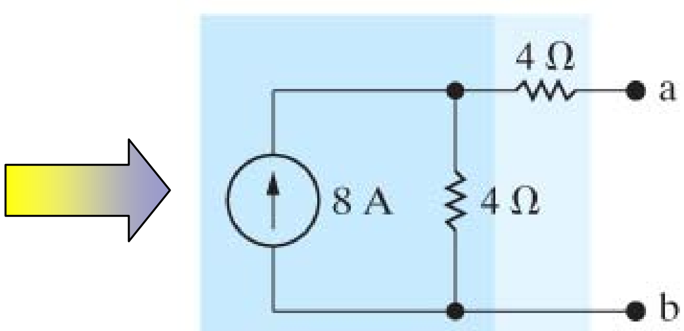


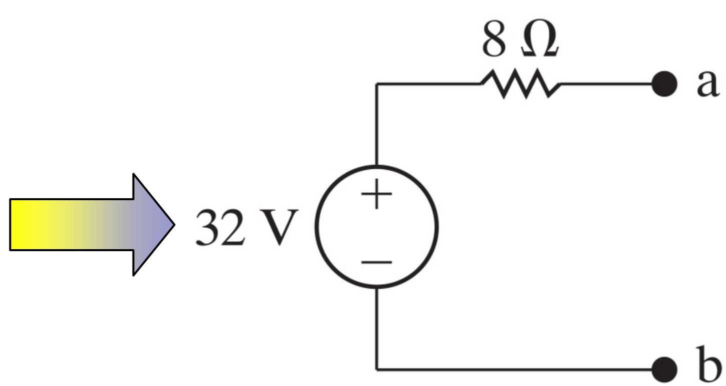
Method 2 to get V_{Th} and R_{Th}

- Use a series of source transformations when the circuit contains only independent sources.





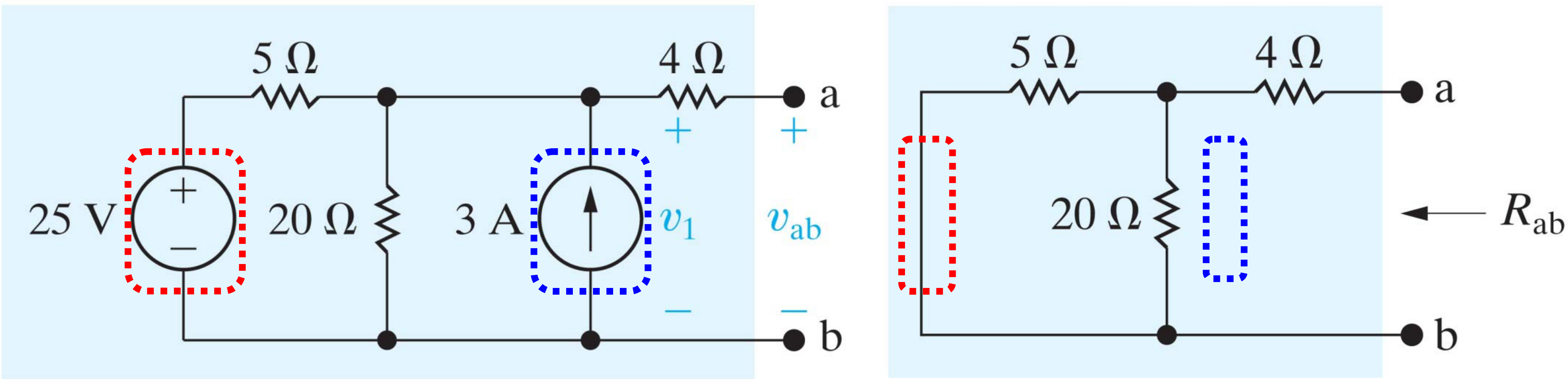




Same as by v_{oc}, i_{sc}.

Method 1 to get R_{Th} alone

- For circuits with only independent sources,
- Step 1: "Deactivate" all sources: (1) Voltage source $\Rightarrow$ Short, (2) Current source $\Rightarrow$ Open.



- Step 2: R_{Th} is the resistance seen by an observer looking into the network at the designated terminal pair.

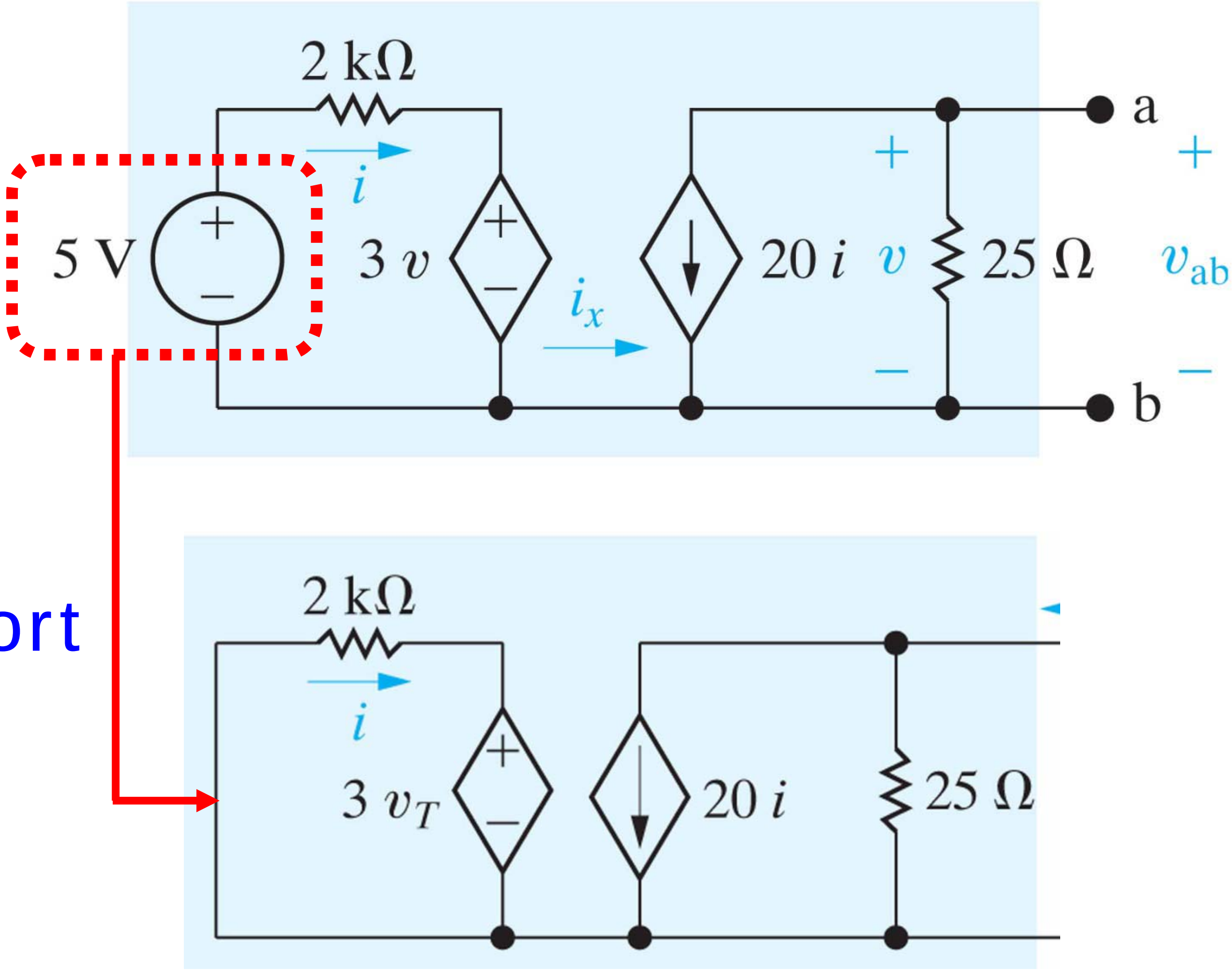
E.g. $R_{Th} = R_{ab} = [(5\ \Omega)//(20\ \Omega)]+(4\ \Omega) = 8\ \Omega$.

Method 2 to get R_{Th} alone

- For circuits with or without dependent sources:
- Step 1: Deactivate all independent sources;
- Step 2: Apply a test voltage v_T or test current i_T source to the designated terminals;
- Step 3: Calculate the terminal current i_T if a test voltage v_T source is used and vice versa;
- Step 4: Get the Thévenin resistance by $R_{Th} = \frac{v_T}{i_T}$.

Example 4.11: Find R_{Th} of a circuit (1)

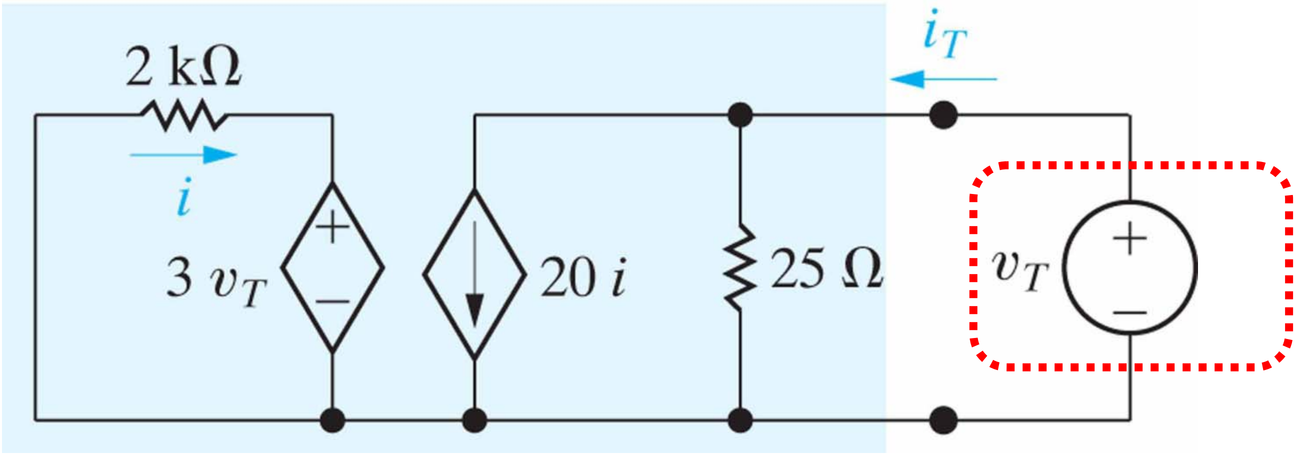
- Step 1: Deactivate all independent sources:



Short

Example 4.11 (2)

- Step 2: Assume a test voltage source v_T:

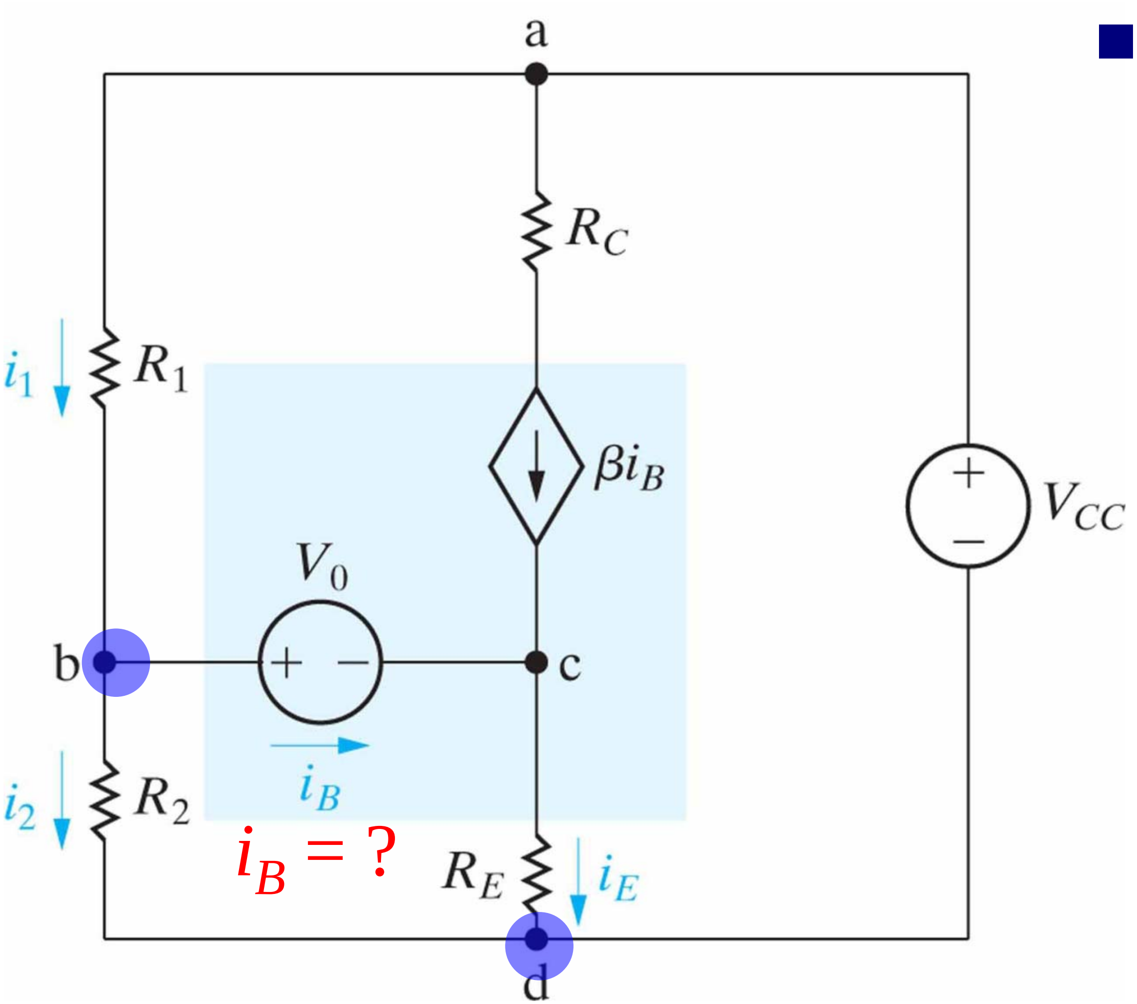


- Step 3: Calculate terminal current i_T ($\propto v_T$):

$$i_T = \frac{v_T}{25\,\Omega} + 20\left(\frac{-3v_T}{2\,\text{k}\Omega}\right) = \frac{v_T}{100\,\Omega}.$$

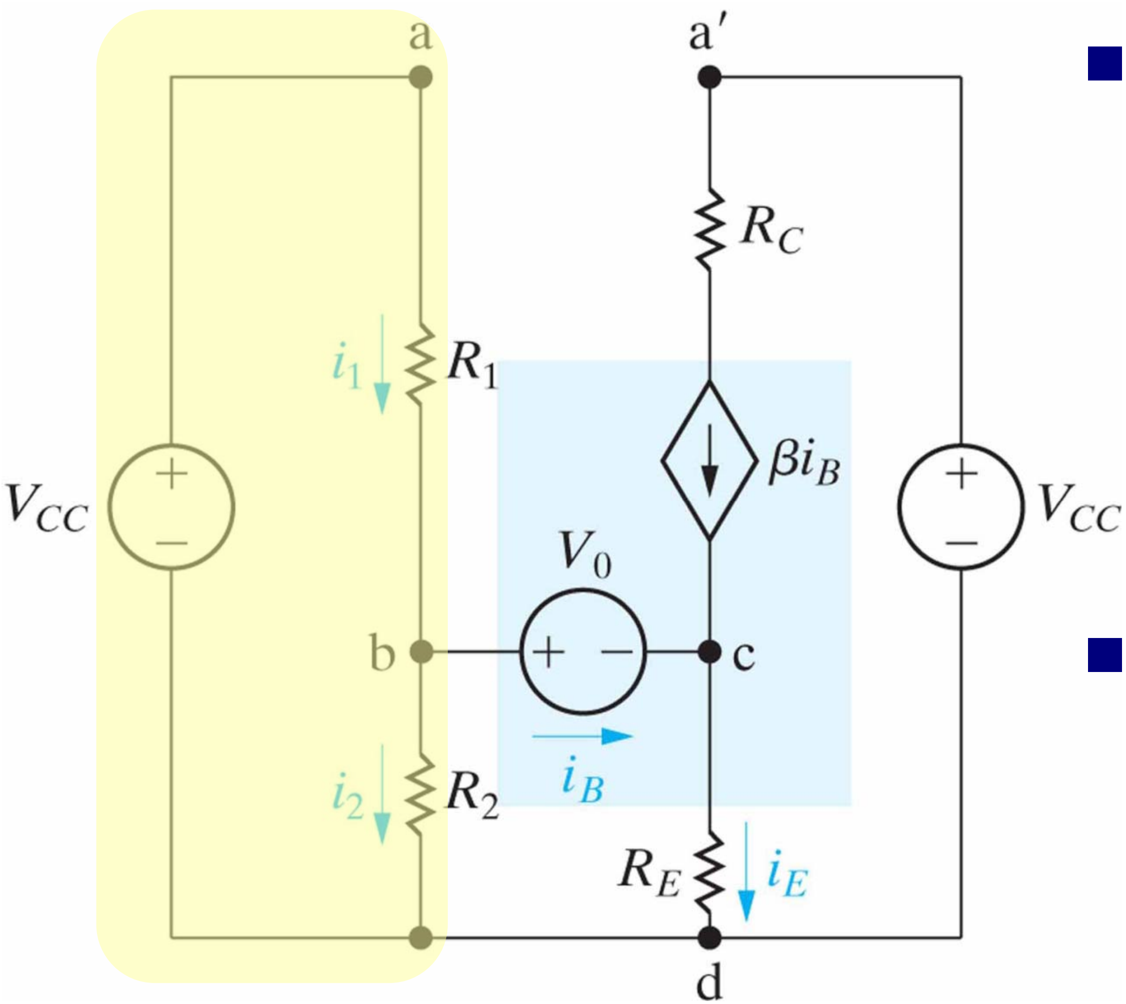
- Step 4: $R_{Th} = v_T / i_T = 100\ \Omega$.

Amplifier circuit solved by equivalent circuit (1)



- To find the equivalent circuit that "drives" terminals b and d (the input of a BJT transistor), we can redraw the circuit as if it were composed of two stages.

Amplifier circuit solved by equivalent circuit (2)



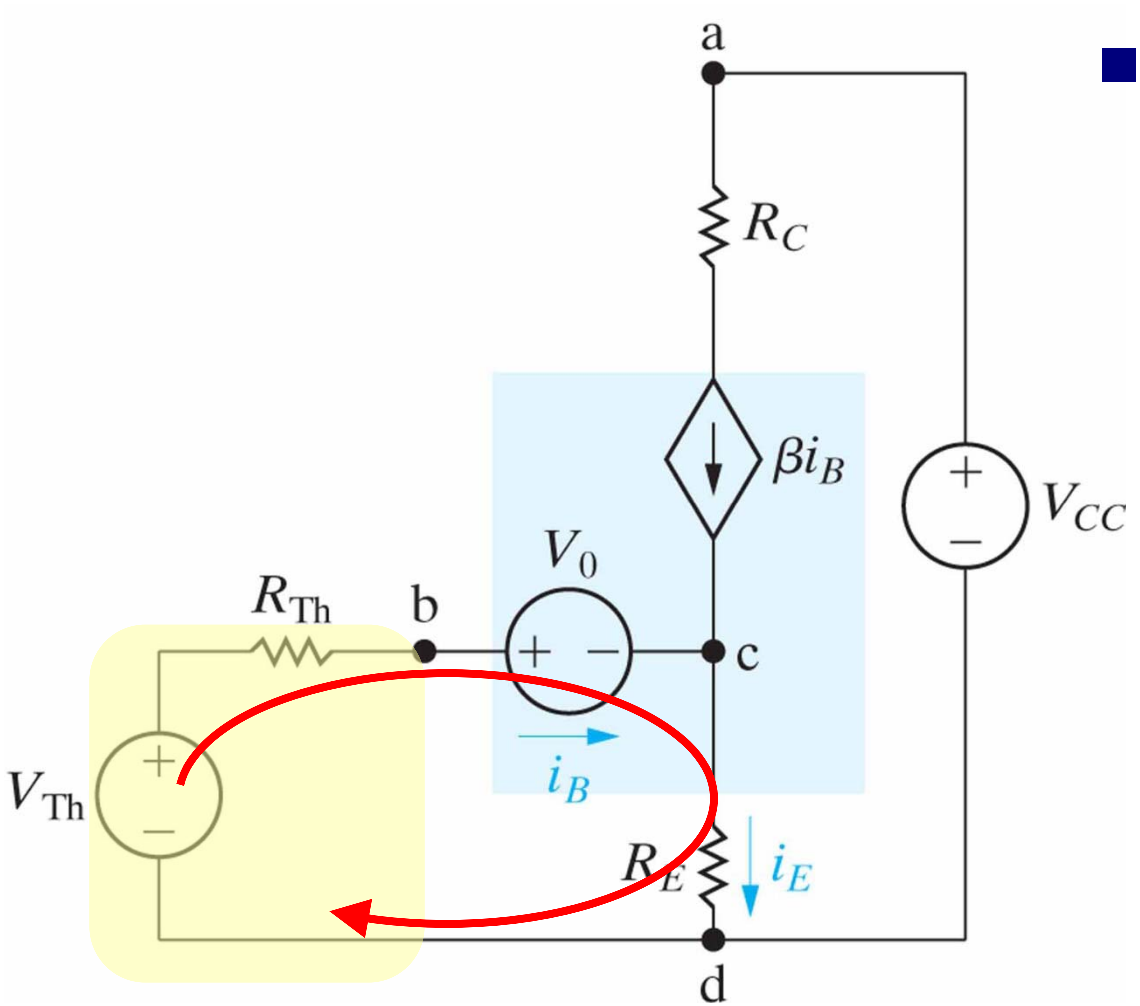
- For terminals b, d, the left part has an open-circuit voltage:

$$v_{Th} = \frac{R_2}{R_1 + R_2} v_{CC},$$

- The resistance with V_{CC} gets shorted is:

$$R_{Th} = R_1 // R_2 = \frac{R_1 R_2}{R_1 + R_2}$$

Amplifier circuit solved by equivalent circuit (3)



- The left part can be replaced by a Thévenin circuit. With this, we can apply KVL to loop bcdb:

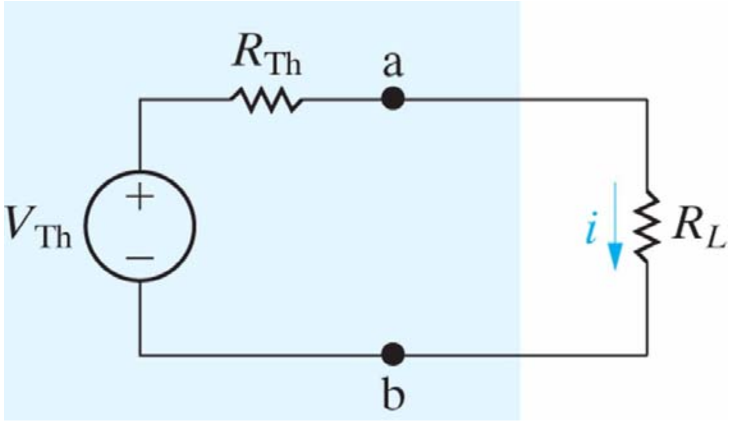
$$v_{Th} = R_{Th} i_B + V_0 + (1+\beta) i_B R_E,$$

$$i_B = \frac{V_{Th} - V_0}{R_{Th} + (1+\beta) R_E}.$$

Section 4.12
Maximum Power Transfer

Formulation (1)

- Consider a circuit (represented by a Thévenin equivalent) loaded with a resistance R_L. The power dissipation at R_L is:



$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L,$$

- To find the value of R_L that leads to maximum power transfer, perform derivative:

$$\frac{dp}{dR_L} = V_{Th}^2 \left[\frac{(R_{Th} + R_L)^2 - 2R_L(R_{Th} + R_L)}{(R_{Th} + R_L)^4} \right].$$

Formulation (2)

- When the derivative equals zero, p is maximized:

$$\frac{dp}{dR_L} = V_{Th}^2 \left[\frac{(R_{Th}+R_L)^2 - 2R_L(R_{Th}+R_L)}{(R_{Th}+R_L)^4} \right] = 0,$$

$$\Rightarrow (R_{Th}+R_L)^{\cancel{2}} - 2R_L\cancel{(R_{Th}+R_L)} = 0,$$

$$\Rightarrow R_{Th}+R_L = 2R_L,$$

$$\Rightarrow \boxed{R_L = R_{Th}}$$

$\Rightarrow$ The maximum transfer power is:

$$p_{\max} = \left(\frac{V_{Th}}{R_{Th}+R_{Th}} \right)^2 R_{Th} = \frac{V_{Th}^2}{4R_{Th}}.$$

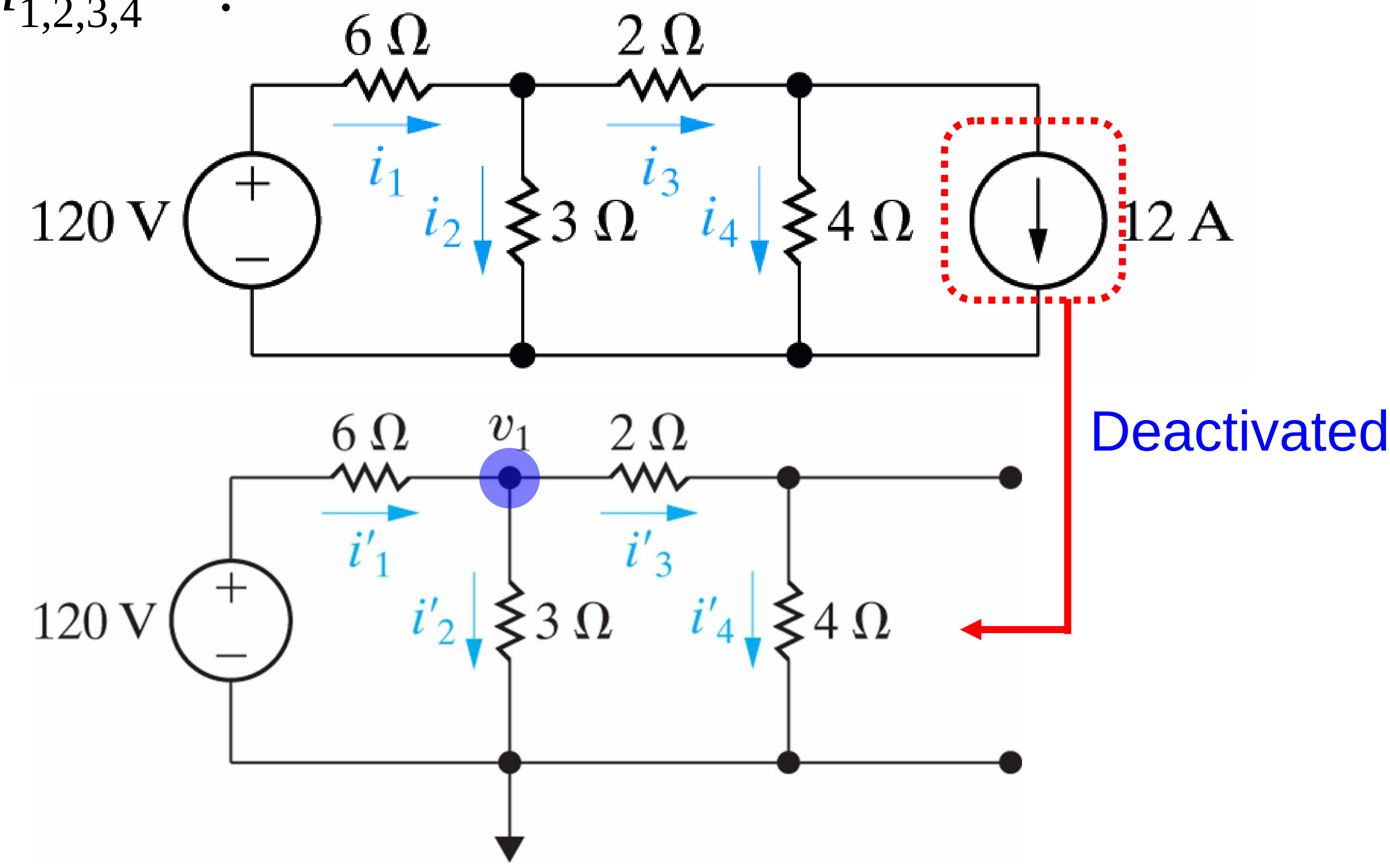
Section 4.13 Superposition

What is superposition?

- In a circuit consisting of linear elements only, superposition allows us to activate one independent source at a time and sum the individual voltages and currents to determine the actual voltages and currents when all independent sources are active.
- Superposition is useful in designing a large system, where the impact of each independent source is critical for system optimization.

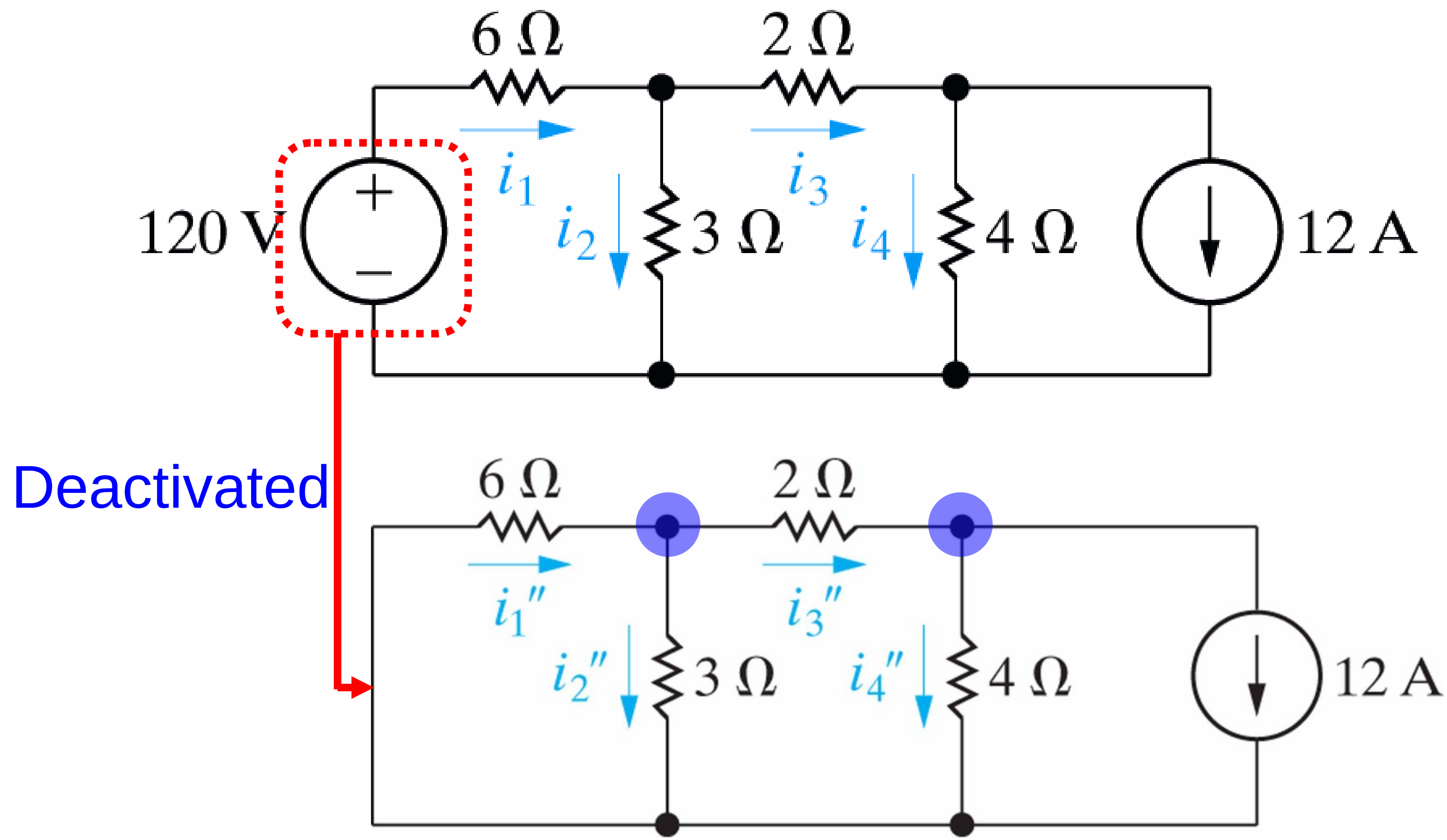
Example (1)

- Q: $i_{1,2,3,4} = ?$



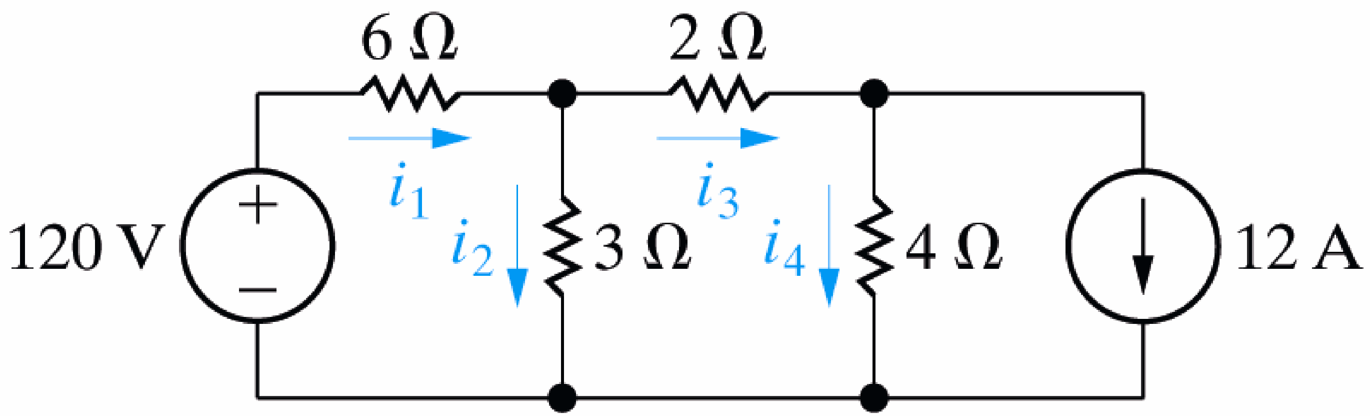
$$\Rightarrow \{i'_1 = 15\text{ A},\ i'_2 = 10\text{ A},\ i'_3 = i'_4 = 5\text{ A}\}.$$

Example (2)



$$\Rightarrow \{i_1'' = 2\text{ A},\ i_2'' = -4\text{ A},\ i_3'' = 6\text{ A},\ i_4'' = -6\text{ A}\}.$$

Example (3)



$$\Rightarrow \begin{cases} i_1 = i_1' + i_1'' = \boxed{15} + 2 = 17\text{ A}, \\ i_2 = i_2' + i_2'' = \boxed{10} - 4 = 6\text{ A}, \\ i_3 = i_3' + i_3'' = 5 + \boxed{6} = 11\text{ A}, \\ i_4 = i_4' + i_4'' = 5 - \boxed{6} = -1\text{ A}, \end{cases}.$$

- The 12-A (120-V) source is more important in determining i_1, i_2, (i_3, i_4).

Key points

- How to solve a circuit by the Node-Voltage Method and Mesh-Current Method systematically?
- What is the meaning of equivalent circuit? Why is it useful?
- How to get the Thévenin equivalent circuit?
- What is superposition? Why is it useful?