# Lecture 8: Mechanical Vibration

- Discrete systems
  - Energy method
  - Lumped-parameter analysis
    - » 1 d.o.f.
    - » Multi-d.o.f. (Eigenvalue analysis)
- Continuous systems
  - Direct solving of partial differential equations
  - Rayleigh's method (the energy approach)
- Example: a laterally-driven folded-flexure comb-drive resonator

Reference: Singiresu S. Rao, Mechanical Vibrations, 2nd Ed., Addison-Wesley Publishing Company, Inc., 1990



# Energy Method

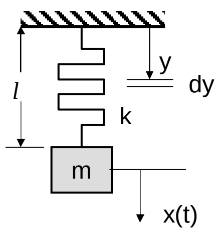
- Conservation of energy; the maximum kinetic energy is equal to the maximum potential energy: $T_{max} = V_{max}$
- Also known as Rayleigh's energy method
- Example: Effect of spring mass $m_s$ on the resonant frequency $\omega_n$

**Kinetic energy of spring length dy:**

$$dT_s =$$

**Total kinetic energy:**

$$T =$$



## *Cont'd*

- **The total potential energy:**

$$U = \frac{1}{2} k x^2$$

- **By assuming a harmonic motion $x(t) = X \cdot \cos\omega_n t$,**

$$T_{max} = \frac{1}{2}\left(m + \frac{m_s}{3}\right) X^2 \omega_n^2$$

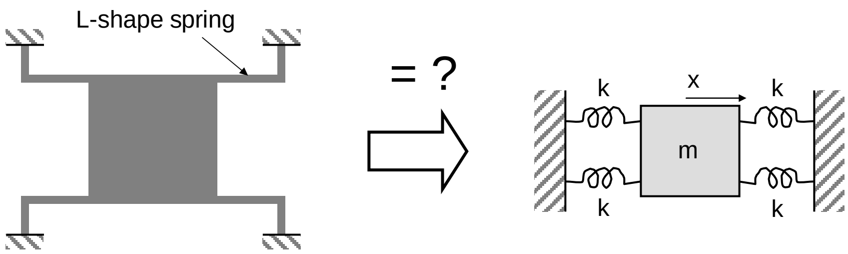
$$U_{max} = \frac{1}{2} k X^2$$

- **By equating $T_{max} = V_{max}$,**

$$\omega_n = \sqrt{\frac{k}{m + m_s / 3}}$$

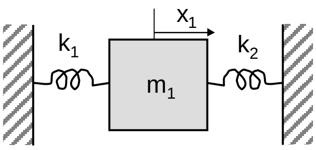


## *Lumped-Parameter Model*

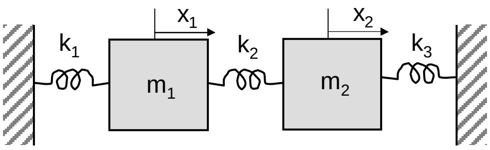



- **Simplified description of 3D physical model using minimum required number of variables (coordinates)**
  - **Do we have "mass-less" spring? A valid assumption?**
- **Can consist of a set of ordinary differential equations depending on the number of variables**
  - **In "Linear Control Systems", we call them the state-space equations**

## Degree of Freedom




1 degree of freedom system

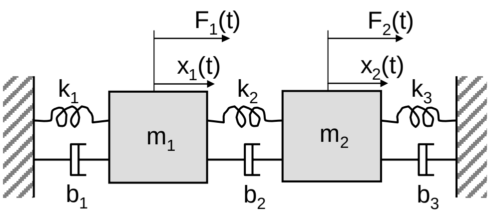



2 degree of freedom system

- **The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time defines the degree of freedom of the system**



## Equations of Motion for a 2 D.O.F. System




$m_1\ddot{x}_1 =$

$m_2\ddot{x}_2 =$

## *Equations of Motion for a 2 D.O.F. System*

$$[m]\ddot{\vec{x}} + [b]\dot{\vec{x}} + [k]\vec{x} = \vec{F}$$

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, [b] = \begin{bmatrix} b_1 + b_2 & -b_2 \\ -b_2 & b_2 + b_3 \end{bmatrix}, [k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$$F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

**In addition to the free-body diagram, equation of motion can also be derived through the Lagrange's equation from the energy perspective**



## *Solving of Dynamic Equation*

- **Gives complete transient response under**
  - **Free vibration: without external applied force**
    - » **How can a structure move without a force?**
    - » **Natural frequency and damped natural frequency can be obtained**
  - **Forced vibration: with external applied force**
- **Motion Types:**
    - » **Underdamped**
    - » **Critical damped**
    - » **Overdamped**
- **Remember how to solve a set of linear ordinary differential equations for multiple d.o.f. systems?**

## Determine Resonant Frequency

- Design of micromechanical devices needs to know natural frequency and damping
  - To many performance indexes of the transient response, such as rise time, overshoot, and settling time
- Resonant frequencies of a lumped-parameter mechanical system can be obtained by
  - Solving the eigenvalue problem (exact solution)
  - Rayleigh's Method (approximate solution)
  - etc



## Eigenvalue Problem

- Under free vibration and no damping, natural frequencies of a multi-d.o.f system are solutions of the eigenvalue problem

$$Let\ \vec{x} = \bar{x}\, sin(\omega t), then$$

$$[[K] - \omega^2 [M]]\bar{x} = \vec{0}$$

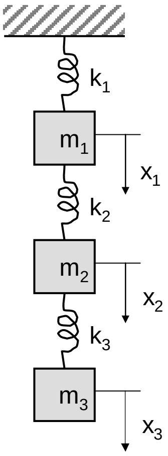
$$\bar{x} \neq 0, \Rightarrow \Delta = [[K] - \omega^2 [M]] = 0$$

$$\times [K]^{-1} \Rightarrow [[I] - \omega^2 [K]^{-1} [M]] = 0, [[I] - \underbrace{\frac{m\omega^2}{k}}_{\alpha} [D]] = 0$$

- The roots $\alpha_i = m\omega_i^2/k$, so $\omega_i$ can be solved
- The eigenvector corresponding to the individual eigenvalue is the mode shape of the system

## *Example*

**From the free-body diagram:**






## *Cont'd*

- **Let $m_1 = m_2 = m_3 = m$, $k_1 = k_2 = k_3 = k$, and $\omega = \sqrt{(k/m)}$:**

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [0]$$

$$\Rightarrow m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [0]$$

$$\Rightarrow \left| k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \omega^2 m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| I - \underbrace{\frac{m\omega^2}{k}}_{\alpha}\underbrace{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \right| = 0$$

## Cont'd

- $\alpha_i = m\omega_i^2/k$, solve:

$$\alpha_1 = \frac{m\omega_1^2}{k} = 0.19806, \omega_1 = 0.44504\sqrt{\frac{k}{m}}$$

$$\alpha_2 = \frac{m\omega_2^2}{k} = 1.55530, \omega_2 = 1.2471\sqrt{\frac{k}{m}}$$

$$\alpha_3 = \frac{m\omega_3^2}{k} = 3.24900, \omega_3 = 1.8025\sqrt{\frac{k}{m}}$$



## Cont'd: Mode Shapes

- For each solved $\omega_i$, recall that:

$$[[K] - \omega_i^2 [M]]\vec{x} = \vec{0}$$

$$\left[ k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \left(\frac{\alpha_i k}{m}\right) \cdot m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x_1^i \\ x_2^i \\ x_3^i \end{bmatrix}$$

$$= k \left[ \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \alpha_i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} x_1^i \\ x_2^i \\ x_3^i \end{bmatrix} = 0$$

- We can solve the eigenvector $x_j^i$ with respect to each $\alpha_i$

## Cont'd: Mode Shapes

- 1st mode, $\alpha_1 = 0.19806$

$$\vec{x}^1 = \vec{x}_1^1 \begin{Bmatrix} 1.0 \\ 1.8019 \\ 2.2470 \end{Bmatrix}$$

- 2nd mode, $\alpha_2 = 1.5553$

$$\vec{x}^2 = \vec{x}_1^2 \begin{Bmatrix} 1.0 \\ 0.4450 \\ -0.8020 \end{Bmatrix}$$

- 3rd mode, $\alpha_3 = 3.2490$

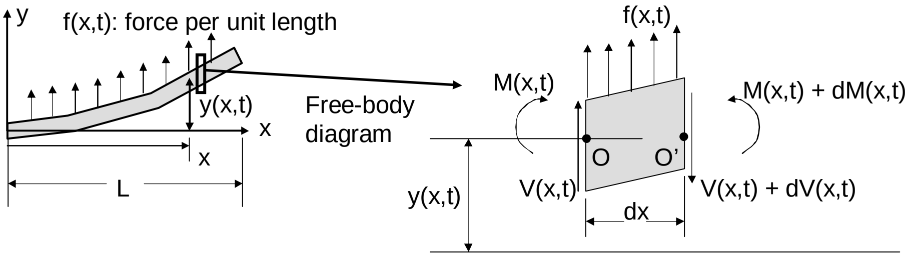
$$\vec{x}^3 = \vec{x}_1^3 \begin{Bmatrix} 1.0 \\ -1.2468 \\ 0.5544 \end{Bmatrix}$$



## Vibration of Continuous Systems

- A system of infinite degrees of freedom
- The equation of motion may be described by a partial differential equation which can be solved by the method of separation of variables
- Many methods can be used to find approximate resonant frequencies and mode shapes (e.g. the Rayleigh's method)

## Example: Lateral Vibration of Beams

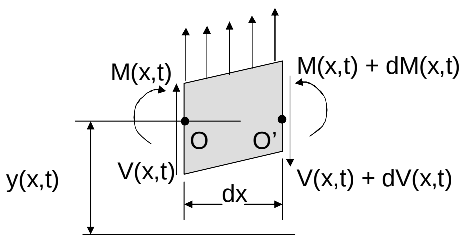



- **What is the dynamic equation?**
- **The inertia force (i.e. *f* = *ma*):**



## Example: the Lateral Vibration of Beams

- **The sum of moments around the point *O* is ZERO**




- **Substitute $V = \partial M/\partial x$ into the last equation:**

$$-\frac{\partial^2 M(x,t)}{\partial x^2} + f(x,t) = \rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2}$$

$$-\frac{\partial^2}{\partial x^2}[\qquad\qquad] + f(x,t) = \rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2}$$

**For a uniform beam:** $EI\frac{\partial^4 y(x,t)}{\partial x^4} + \rho A\frac{\partial^2 y(x,t)}{\partial t^2} = f(x,t)$

## *Example: Lateral Vibration of Beams*

$$EI\frac{\partial^4 y(x,t)}{\partial x^4}+\rho A\frac{\partial^2 y(x,t)}{\partial t^2}=0$$

- **For free vibration, $f(x,t) = 0$, we require**
  - **Two initial conditions, for example:**
    - » $y(x, t = 0) = y_o(x) = 0$
    - » $\partial y/\partial t|_{(x,\, t = 0)} = 0$
  - **Four boundary conditions, for example:**
    - » **Free end**
      - – Bending moment = $EI(\partial^2 y/\partial x^2) = 0$
      - – Shear force = $EI\partial^3 y/\partial x^3 = 0$
    - » **Simply supported (pinned) end**
      - – Deflection $y = 0$
      - – Bending moment = $EI(\partial^2 y/\partial x^2) = 0$
    - » **Clamped end**
      - – Deflection $y = 0$
      - – Slope $\partial y/\partial x = 0$

**We will use these two b.c.'s to solve for the Fixed-pinned beam**



## *Solve Lateral Vibration of Beams*

- **Use the method of separation of variables $y(x,t) = Y(x)\cdot T(t)$**

$$EIT(t)\frac{d^4Y(x)}{dx^4}+\rho AY(x)\frac{d^2T(t)}{dt^2}=0$$

$$\frac{EI/\rho A}{Y(x)}\frac{\partial^4 Y(x)}{\partial x^4}=-\frac{1}{T(t)}\frac{d^2T(t)}{dt^2}=a=\omega^2$$

(3)

(2)

## *Solve the Lateral Vibration of Beams*

- ***Y(x)*** **can be solved as:**

$$Y(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x}$$

**Or,**

$$Y(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 \cosh\beta x + C_4 \sinh\beta x$$

**The natural frequencies of the beam are (from (1)):**

$$\omega = \beta^2 \sqrt{\frac{EI}{\rho A}} = (\beta l)^2 \sqrt{\frac{EI}{\rho A l^4}}$$

**The β*l* product depends on the boundary conditions**



## *Solve Lateral Vibration of a Fixed-Pinned Beam*

- **Four B.C.'s for a fixed-pinned beam are substituted into Y(x):**

$$Y(0) = 0 \longrightarrow C_1 + C_3 = 0 \Rightarrow C_3 = -C_1$$

$$\frac{dY}{dx}(0) = 0 \longrightarrow \beta(C_2 + C_4) = 0 \Rightarrow C_4 = -C_2$$

$$\therefore Y(x) = C_1(\cos\beta x - \cosh\beta x) + C_2(\sin\beta x - \sinh\beta x) \quad (4)$$

$$Y(l) = 0 \longrightarrow C_1(\cos\beta l - \cosh\beta l) + C_2(\sin\beta l - \sinh\beta l) = 0 \quad (5)$$

$$\underbrace{EI\frac{d^2Y}{dx^2}(l)}_{0} = 0 \longrightarrow -C_1(\cos\beta l + \cosh\beta l) - C_2(\sin\beta l + \sinh\beta l) = 0 \quad (6)$$

**So,**

$$\begin{bmatrix} \cos\beta l - \cosh\beta l & \sin\beta l - \sinh\beta l \\ -(\cos\beta l + \cosh\beta l) & -(\sin\beta l + \sinh\beta l) \end{bmatrix} = 0 \quad (7)$$

## *Cont'd*

- **From the last matrix, we get the determinant:**

$$tan\beta l = tanh\beta l$$

- **The many roots of this equation, $\beta_n l$, will define the natural frequencies:**

$$\omega_n = (\beta_n l)^2 \sqrt{\frac{EI}{\rho A l^4}}$$

- **Mode shape: $Y_n(x)$, $Y(x)$, $y_n(x,y)$, and $y(x,t)$:**

$$C_{2n} = -C_{1n}\left(\frac{cos\beta_n l - cosh\beta_n l}{sin\beta_n l - sinh\beta_n l}\right), \quad from\ (5)$$

$$Y_n(x) = C_{1n}\left[(cos\beta_n x - cosh\beta_n x) - \left(\frac{cos\beta_n l - cosh\beta_n l}{sin\beta_n l - sinh\beta_n l}\right)(sin\beta_n x - sinh\beta_n x)\right], \ from\ (4)$$

$$y_n(x,t) = Y_n(x)(A_n cos\,\omega_n t + B_n sin\,\omega_n t)$$

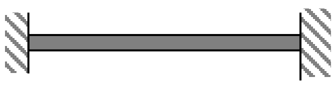
$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t), \quad The\ final\ mode\ shape$$



## *Results of $\beta_n l$ for Various Beam Constraints*

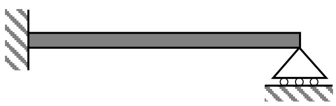
(1) Cantilever beam

$\beta_1 l$ = 1.875104
$\beta_2 l$ = 4.694091
$\beta_3 l$ = 7.854757
$\beta_4 l$ = 10.99541



(2) Doubly-clamped beam

$\beta_1 l$ = 4.730041
$\beta_2 l$ = 7.853205
$\beta_3 l$ = 10.995608
$\beta_4 l$ = 14.137165



(3) fixed-pinned beam

$\beta_1 l$ = 3.926602
$\beta_2 l$ = 7.068583
$\beta_3 l$ = 10.210176
$\beta_4 l$ = 13.351768

## Rayleigh's Method

- An approximate analysis using the energy perspective to find the fundamental natural frequency of continuous systems
- The kinetic energy of a beam:

$$T =$$

- Assume a harmonic variation y(x,t) = Y(x)·cos(ωt), the maximum kinetic energy:

$$T_{\max} = \frac{\omega^2}{2}\int_0^l Y^2(x)\rho A(x)dx$$



## Cont'd

- The potential energy V of a beam: (neglecting the work done by the shear forces)

- The maximum value of *y*(*x,t*) is Y(x), so the maximum potential energy:

$$V_{max} = \frac{1}{2}\int_0^l EI(\frac{d^2Y(x)}{dx^2})^2 dx$$

# *Rayleigh's Method*

- By equating $T_{max}$ to $V_{max}$, we obtain:

$$\omega^2 = \frac{\int_0^l EI\left(\frac{d^2Y(x)}{dx^2}\right)^2 dx}{\int_0^l \rho A Y^2(x) dx}$$

- For example, a stepped beam with various cross sections:

$$\omega^2 = \frac{\int_0^{l_1} E_1 I_1\left(\frac{d^2Y(x)}{dx^2}\right)^2 dx + \int_{l_1}^{l_2} E_2 I_2\left(\frac{d^2Y(x)}{dx^2}\right)^2 dx + \cdots}{\int_0^{l_1} \rho A_1 Y^2(x) dx + \int_{l_1}^{l_2} \rho A_2 Y^2(x) dx + \cdots}$$

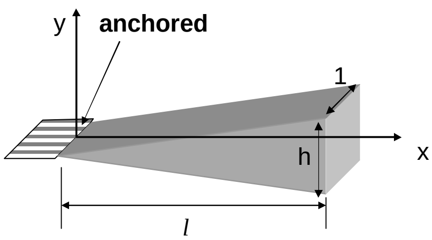
- Where is Y(x) from? You have to choose Y(x), and make sure: (1) it is a reasonable beam deflection curve; (2) Y(x) must satisfy the beam boundary conditions



# *Example: Find the Resonant Frequency*

- Use the deflection curve $Y(x) = (1 - x/l)^2$
  - The cross section $A(x) = hx/l$
  - The moment of inertia $I(x) = 1\cdot(hx/l)^3/12$
- By equating $T_{max}$ to $V_{max}$

$\omega^2 =$

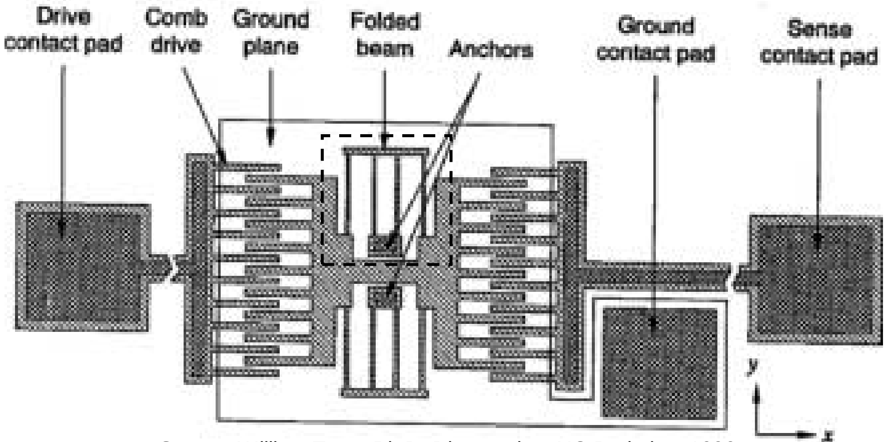



- The exact frequency is (for comparison):

$$\omega = 1.5343\left(\frac{Eh^2}{\rho l^4}\right)^{1/2}$$

## *Lateral Folded-flexure Comb-Drive Resonator*

- What is the resonant frequency of the resonator?
  - A lumped-parameter model would be used for analysis

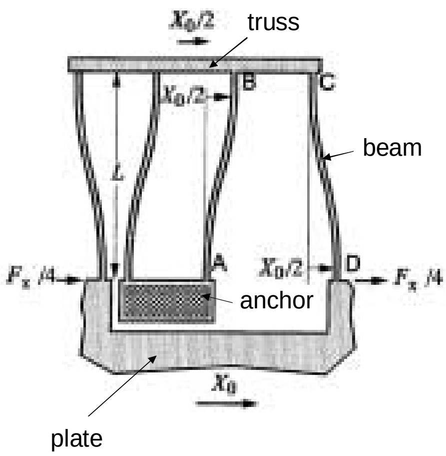



Source: William Tang, Ph.D. Dissertation, UC Berkeley, 1990



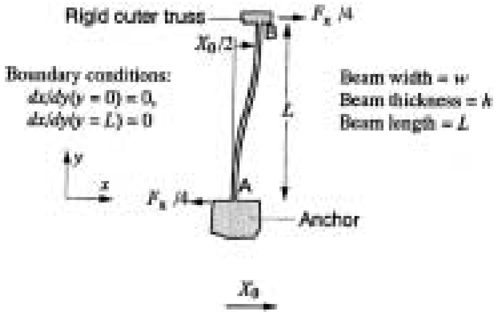
## *Spring Constant $k_x$*

- When the resonant plate moves $X_o$ under a given force $F_o$, the point B and D moves $X_o/2$, respectively
  - The force acting on each beam is $F_o/4$
- The slope at both ends of the beams are identically zero



## *Cont'd*

- **The deflection curve of beam AB is:**






## *Lateral Resonant Frequency*

- **By Rayleigh's energy method:**

$$K.E._{max} = P.E._{max}$$

$$K.E._{max} = K.E._{plate} + K.E._{truss} + K.E._{beam}$$

$$= \frac{1}{2} M_p v_p^2 + \frac{1}{2} M_t v_t^2 + \frac{1}{2} \int v_b^2 dM_b$$

$$=$$

- **For the beam segment AB, remember that:**

$$x_{AB}(y) = \frac{(F_x/4)}{12EI_z}(3Ly^2 - 2y^3) \quad for\ 0 \le y \le L$$

$$x_{AB}(L) = X_o/2 = \frac{F_x L^3}{48EI_z}$$

$$\Rightarrow x_{AB}(y) = \frac{X_o}{2}\left[3\left(\frac{y}{L}\right)^2 - 2\left(\frac{y}{L}\right)^3\right]$$

# Cont'd

- **So the velocity profile for segment AB (multiply ω) is:**

$$v_{AB}(y)=\frac{X_o\omega}{2}\left[3\left(\frac{y}{L}\right)^2-2\left(\frac{y}{L}\right)^3\right]$$

- **The K.E. for beam AB is:**

$$K.E.|_{AB}=\frac{1}{2}\int_0^L \underbrace{\frac{(X_o\omega)^2}{4}\left[3\left(\frac{y}{L}\right)^2-2\left(\frac{y}{L}\right)^3\right]^2}_{v_{AB}^2} dM_{AB}$$

$$=\frac{X_o^2\omega^2 M_{AB}}{8L}\int_0^L\left[3\left(\frac{y}{L}\right)^2-2\left(\frac{y}{L}\right)^3\right]^2 dy \quad \left(dM_{AB}=\frac{M_{AB}}{L}\cdot dy\right)$$

$$=\frac{13}{280}X_o^2\omega^2 M_{AB}$$



# Cont'd

- **Similarly for beam CD, the deflection curve is:**

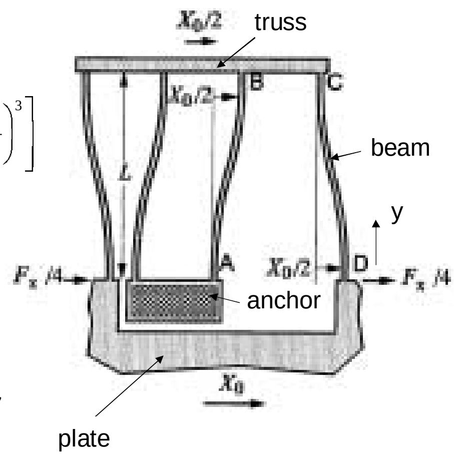
$$x_{CD}(y)=X_o+(-x_{AB}(y))=X_o\left[1-\frac{3}{2}\left(\frac{y}{L}\right)^2+\left(\frac{y}{L}\right)^3\right]$$

- **The velocity profile and K.E. for segment CD are:**

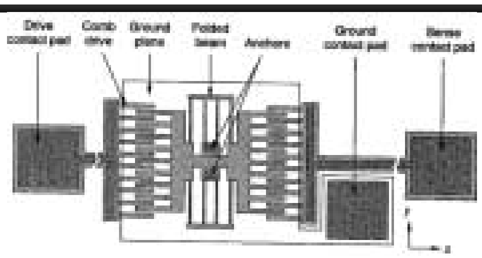
$$v_{CD}(y)=X_o\cdot\omega\left[1-\frac{3}{2}\left(\frac{y}{L}\right)^2+\left(\frac{y}{L}\right)^3\right]$$

$$K.E.|_{CD}=\frac{X_o^2\omega^2 M_{CD}}{2L}\int_0^L\left[1-\frac{3}{2}\left(\frac{y}{L}\right)^2+\left(\frac{y}{L}\right)^3\right]^2 dy$$

$$=\frac{83}{280}X_o^2\omega^2 M_{CD}$$



## *Cont'd: Total Beam Potential Energy*




- **Since,**

$$M_{AB} = M_{CD} = \frac{1}{8} M_b$$

$$\Rightarrow K.E._b = 4 \cdot K.E._{AB} + 4 \cdot K.E._{CD}$$

$$= \frac{13}{560} X_o^2 \omega^2 M_b + \frac{83}{560} X_o^2 \omega^2 M_b$$

$$= \frac{6}{35} X_o^2 \omega^2 M_b$$



## *Cont'd*

- **The total maximum K.E. is**

$$K.E._{\max} = K.E._{plate} + K.E._{truss} + K.E._{beam}$$

$$= X_o^2 \omega^2 \left( \frac{1}{2} M_p + \frac{1}{8} M_t + \frac{6}{35} M_b \right)$$

- **The total maximum P.E. is:**

$$P.E._{\max} = \int_0^{X_o} F_x \cdot dx = \int_0^{X_o} k_x x \cdot dx = \frac{1}{2} k_x X_o^2$$

- **Equating both equations, we obtain the resonant frequency:**

$$\omega = \sqrt{\frac{k_x}{M_p + \frac{1}{4} M_t + \frac{12}{35} M_b}}$$