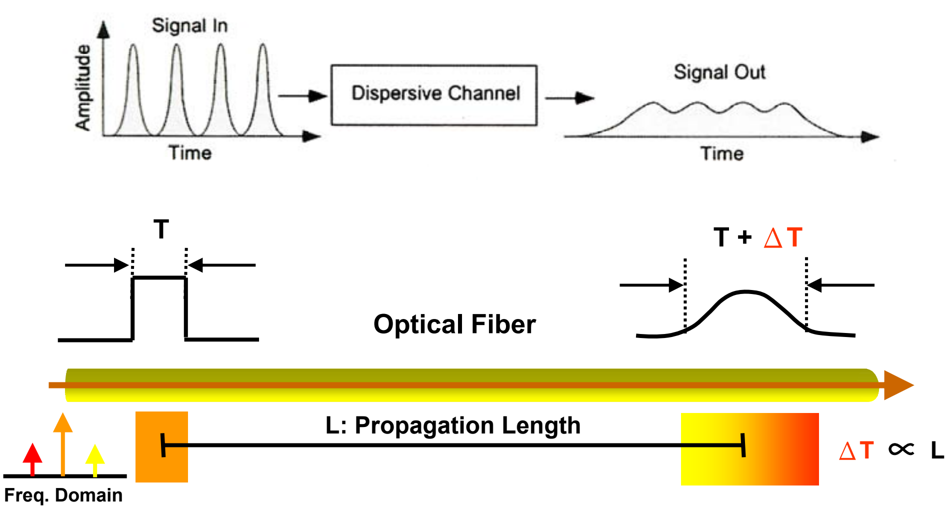
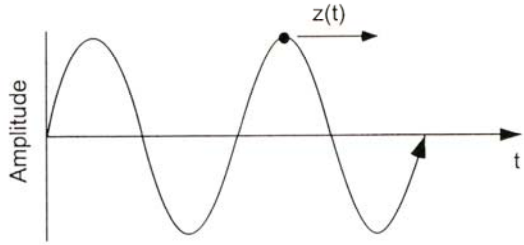
Waveguide Dispersion

Class: Integrated Photonic Devices
Time: Fri. 8:00am ~ 11:00am.
Classroom: 資電206
Lecturer: Prof. 李明昌(Ming-Chang Lee)

Chromatic Dispersion



Phase Velocity



$$k \equiv k_0 n$$

Consider a monochromatic plane wave

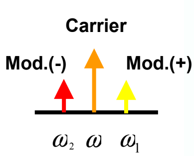
$$\vec{\mathbf{E}}(r,t) = \vec{\mathbf{E}_0}\exp(jkz - j\omega t) \quad \text{and} \quad \phi = kz - \omega t$$

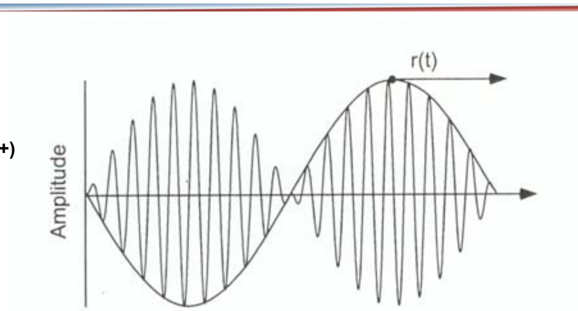
For a point of constant phase; that is

$$z = \frac{\omega}{k}t + \textbf{constant}$$

It is like the point moving with a velocity (v_p) : $\dfrac{\omega}{k} = \dfrac{\omega}{k_0 n} = \dfrac{c}{n}$

Group Velocity





In reality, the light is always be modulated with a pulse. Therefore, there is always a finite bandwidth of frequencies. With the frequency assigned

$$\begin{cases} \omega_1 = \omega + \Delta\omega \quad \text{and} \quad \omega_2 = \omega - \Delta\omega \\ k_1 = k + \Delta k \quad \text{and} \quad k_2 = k - \Delta k \end{cases}$$

Group Velocity

Assuming the waves have equal amplitudes, E_0, the superposition can be described as

$$E_1 + E_2 = E_0(\cos[(k+\Delta k)z - (\omega+\Delta\omega)t)] + \cos[((k-\Delta k)z - (\omega-\Delta\omega)t)])$$

$$\downarrow$$

$$E_1 + E_2 = 2E_0 \underbrace{\cos(kz-\omega t)}_{\text{Carrier}} \underbrace{\cos(\Delta kz - \Delta\omega t)}_{\text{Modulation}}$$

Like phase velocity, for a point of constant phase on the crest of the envelop

$$z = \frac{\Delta\omega}{\Delta k} t + \textbf{constant}$$

It is like the point moving with a velocity: $\frac{\Delta\omega}{\Delta k}$

Group Velocity

The group velocity is defined:

$$v_g = \lim_{\Delta\omega\to 0} \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

Recall $\omega = \frac{kc}{n}$

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(\frac{kc}{n}) = \frac{c}{n} - \frac{kc}{n^2}\frac{dn}{dk}$$

$$= v_p - \frac{kc}{n^2}\frac{dn}{dk}$$ (Relation between v_p abd v_g)

$$\tau_g = v_g^{-1} = \left(\frac{c}{n} - \frac{kc}{n^2}\frac{dn}{dk}\right)^{-1}$$ (Propagation Time per unit length)

Group Velocity

Index vs. group index

Index: $v_p = \frac{c}{n}$

Group Index: $v_g = \frac{c}{N}$ $\boxed{N = n - \lambda_0 \frac{dn}{d\lambda_0}}$

In reality, the group velocity is usually a function of optical frequency

$$\frac{d}{d\omega} v_g^{-1} = \frac{d^2 k}{d\omega^2} \neq 0$$

$\frac{d^2 k}{d\omega^2}$: group-velocity dispersion

In the measurement of the broadening of optical pulse, another dispersion coefficient is defined as

$$D_\lambda = -\frac{2\pi c}{\lambda_0^2} \frac{d^2 k}{d\omega^2}$$ unit: ps/km·nm

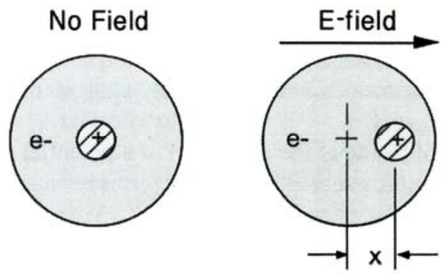
Chromatic Dispersion in Waveguide

- **Material Dispersion**
  - **The refractive index varies with frequency (wavelength)**
- **Model Dispersion**
  - **The propagation constant varied with different mode (m=1,2,3,..)**
- **Waveguide Dispersion**
  - **The propagation constant varied with frequency (wavelength)**

Material Dispersion

Recall $\vec{D}(r,t) = \varepsilon_0 \vec{E}(r,t) + \vec{P}(r,t)$ ← Electric Polarization

Atom



The electric polarization comes from each dipole moment. As in the figure,

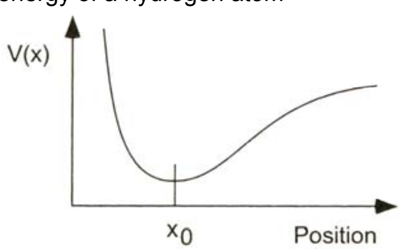
$$\vec{p} = q \cdot 2\vec{x}$$

Electric Polarization is the bulk polarization of the material

$$\vec{P} = N\vec{p}$$

Material Dispersion

The binding energy of a hydrogen atom



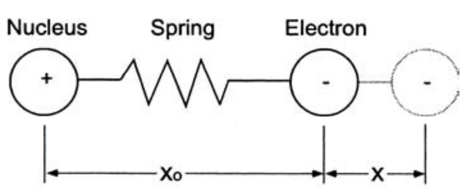
$$V(x) \approx V(x_0) + \frac{1}{2}\frac{d^2V}{dx^2}\bigg|_{x=x_0}(x-x_0)^2$$

$$\downarrow$$

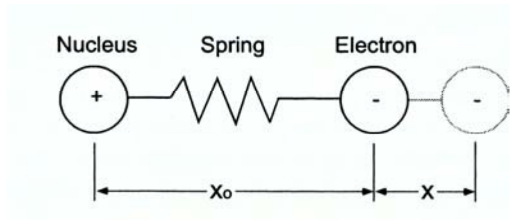
$$V(x) = V_0 + \frac{1}{2}kx^2$$

(Like a spring)!

The Lorentz model of the atom consists of a heavy nucleus bound to a light electron



Material Dispersion



The Lorentz model acts as harmonic oscillator

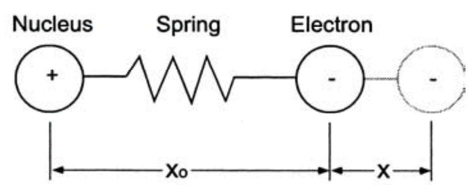
$$\vec{F} = q\vec{E}$$

$$m = \frac{m_e m_n}{m_e + m_n} \approx m_e$$ (We use the center of mass picture)

Therefore, the motion can be modelled as

$$m\frac{d^2x}{dt^2} + \zeta m\frac{dx}{dt} + kx = -eE(t)$$ (recall q = -e)

Material Dispersion



$$m\frac{d^2x}{dt^2} + \zeta m\frac{dx}{dt} + kx = -eE(t)$$

Divided by m

Friction

$$\frac{d^2x}{dt^2} + \zeta\frac{dx}{dt} + \omega_0^2 x = -\frac{e}{m}E(t)$$ where $\omega_0 \equiv \sqrt{\frac{k}{m}}$

Assume $E(t) = E_0 \exp(j\omega t)$ $\boxed{\mathbf{x}(t) = \frac{-e/m}{\omega_0^2 - \omega^2 + j\zeta\omega} E_0 \exp(j\omega t)}$

Material Dispersion

$$\boxed{\mathbf{x}(t) = \frac{-e/m}{\omega_0^{\,2} - \omega^2 + j\zeta\omega} E_0 \exp(j\omega t)}$$

Recall the electric displacement

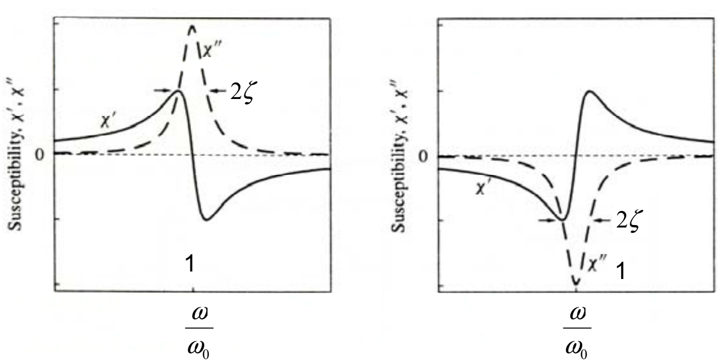
$$\begin{aligned}
\mathbf{D} &= \varepsilon_0 \mathbf{E} + Nq\mathbf{x} = \varepsilon_0 \mathbf{E} - Ne\mathbf{x} \\
&= \varepsilon_0 \mathbf{E} + \frac{Ne^2/m}{\omega_0^{\,2} - \omega^2 + j\xi\omega}\mathbf{E} \\
&= \varepsilon_0 (1 + \underbrace{\frac{Ne^2/m}{\varepsilon_0(\omega_0^{\,2} - \omega^2 + j\xi\omega)}}_{\chi})\mathbf{E}
\end{aligned}$$

χ Susceptibility $\chi(\omega) = \chi'(\omega) - j\chi''(\omega)$

$$\chi'(\omega) = \frac{(Ne^2/m)(\omega_0^{\,2} - \omega^2)}{\varepsilon_0[(\omega_0^{\,2} - \omega^2)^2 + \zeta^2\omega^2]}$$

$$\chi''(\omega) = \frac{(Ne^2/m)(\zeta\omega)}{\varepsilon_0[(\omega_0^{\,2} - \omega^2)^2 + \zeta^2\omega^2]}$$

Material Dispersion



- **χ' represents the dispersion and χ'' represents the loss (or gain).**
- **χ' and χ'' are not independent. Actually, they follow the Kramers-Krönig Relation (KKR) due to the causality.**
- **χ'' > 0 means the loss; χ'' < 0 means the gain.**

Kramers-Krönig Relation (KKR)

$$\chi'(\omega) = \frac{2}{\pi} P \int_0^\infty \frac{\omega' \chi''(\omega')}{\omega'^2 - \omega^2} d\omega' \quad \longleftrightarrow \quad \chi''(\omega) = -\frac{2}{\pi} P \int_0^\infty \frac{\omega' \chi'(\omega')}{\omega'^2 - \omega^2} d\omega'$$

- **Once the real part is known over the entire spectrum, its imaginary part can be found, and vice versa.**
- **$\chi'(\omega)$ is an even function, while $\chi''(\omega)$ is an odd function.**

Material Dispersion

Dielectric Constant: ε

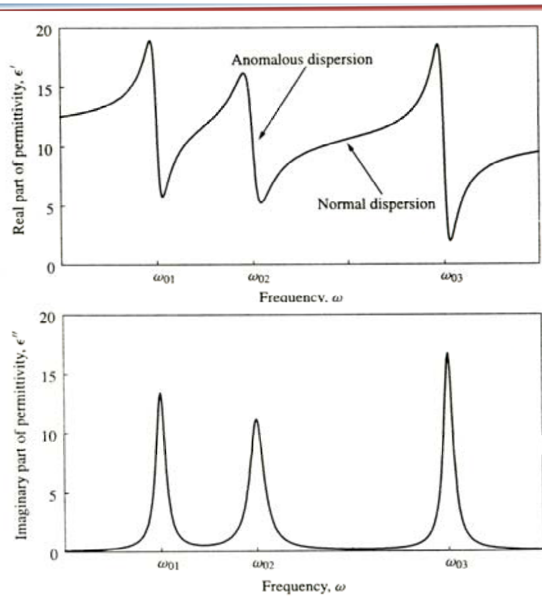
$$\mathbf{D} = \varepsilon_0 (1 + \boldsymbol{\chi})\mathbf{E} = \varepsilon_0 \boldsymbol{\varepsilon} \mathbf{E}$$

$$\boldsymbol{\varepsilon}' = 1 + \chi'$$

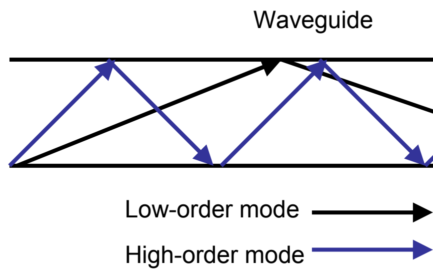
$$\boldsymbol{\varepsilon}'' = \chi''$$

$$\chi = \frac{Ne^2}{\varepsilon_0 m} \sum_n \frac{f_n}{\omega_n^2 - \omega^2 + j\zeta_n \omega}$$

fn: fraction of oscillation strength



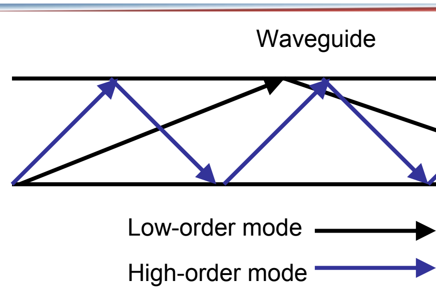
Model Dispersion



- **The propagation time varies with different modes**

$$\Delta\tau = \tau_{low} - \tau_{high}$$

Model Dispersion



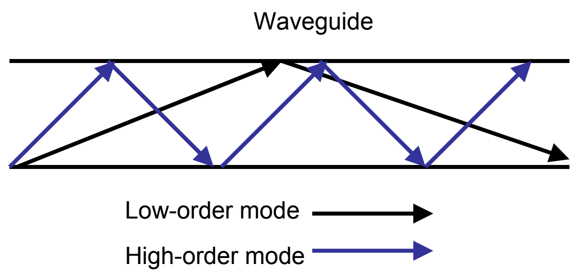
The group delay: $\tau_g = \dfrac{d\beta}{d\omega}$

The fundamental mode: $\beta_f \approx \dfrac{\omega n_{core}}{c}$

The highest-order mode: $\beta_h \approx \dfrac{\omega n_{cladding}}{c}$

$$\longrightarrow \begin{cases} \tau_f = \dfrac{d(\omega n_{core}/c)}{d\omega} = \dfrac{n_{core}}{c} + k_0 \dfrac{dn_{core}}{d\omega} \\ \tau_h = \dfrac{d(\omega n_{cladding}/c)}{d\omega} = \dfrac{n_{cladding}}{c} + k_0 \dfrac{dn_{cladding}}{d\omega} \end{cases}$$

Model Dispersion

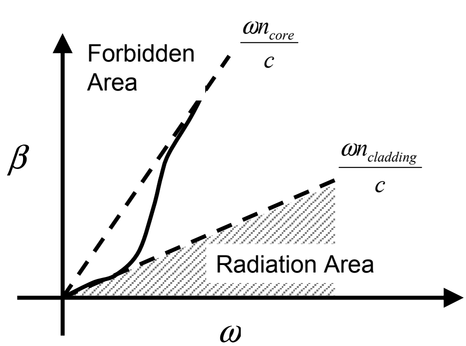


$$\Delta\tau_g = \frac{n_{core} - n_{cladding}}{c} + k_0\left(\frac{dn_{core}}{d\omega} - \frac{dn_{cladding}}{d\omega}\right)$$

$$\approx \frac{n_{core} - n_{cladding}}{c}$$

$$\Delta\tau = \Delta\tau_g L = \frac{(n_{core} - n_{cladding})}{c} L$$

Waveguide Dispersion



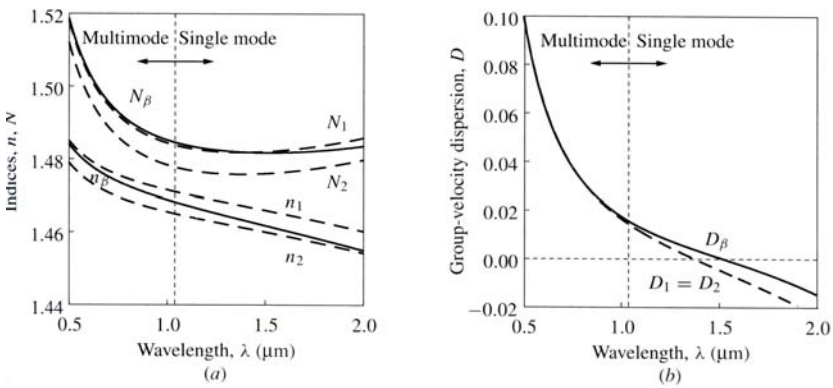
$$\Delta\tau_g = \left.\frac{d\beta}{d\omega}\right|_{\omega_1} - \left.\frac{d\beta}{d\omega}\right|_{\omega_2}$$

$$\approx \frac{d^2\beta}{d\omega^2}\Delta\omega$$

$$\Delta\tau = \frac{d^2\beta}{d\omega^2}\Delta\omega L$$

Group Velocity Dispersion

Combined Waveguide and Material Dispersion



- **The actual single-mode waveguide dispersion should consider both waveguide dispersion and material dispersion**