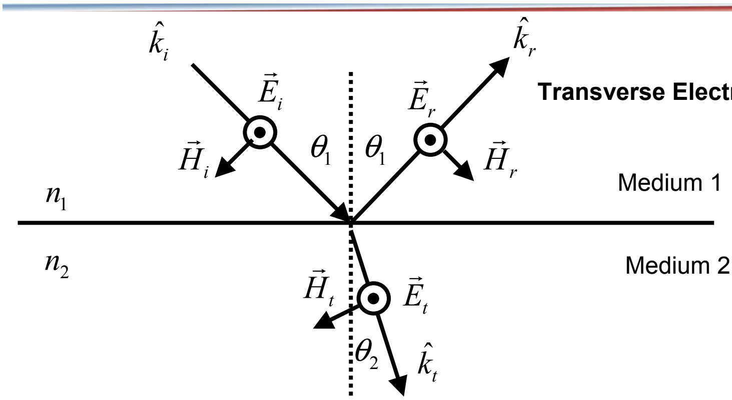
Theory of Optical Waveguide

Class: Integrated Photonic Devices
Time: Fri. 8:00am ~ 11:00am.
Classroom: 資電206
Lecturer: Prof. 李明昌(Ming-Chang Lee)

Reflection and Refraction at an Interface (TE)

Transverse Electric (TE) or s wave

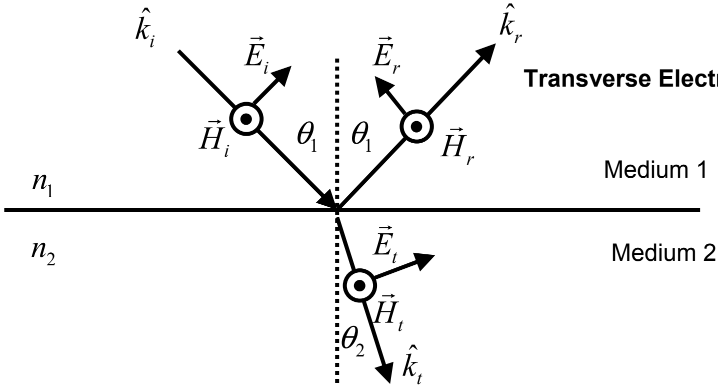


Snell's law: $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

Refraction Coefficient: $r_{TE} \equiv \frac{E_r}{E_i} = \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_i + n_2 \cos\theta_2}$

Reflection and Refraction at an Interface (TM)

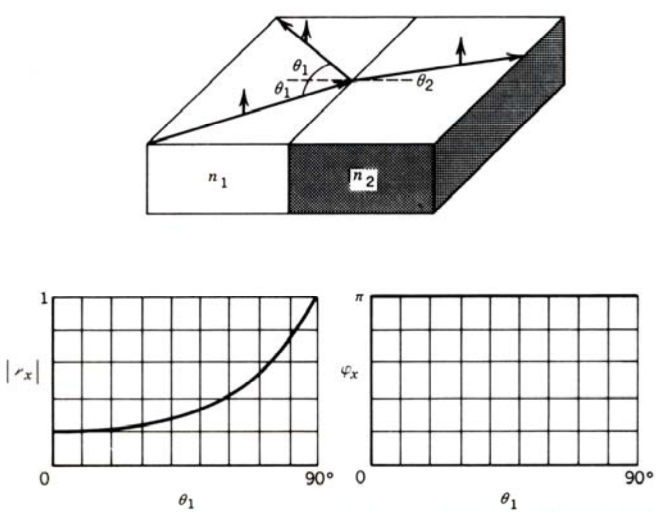
Transverse Electric (TM) or p wave



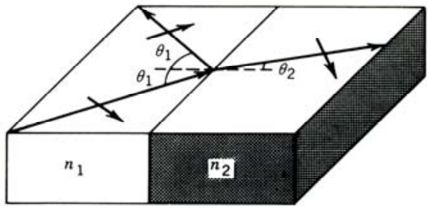
Snell's law: $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

Refraction Coefficient: $r_{TM} \equiv \dfrac{E_r}{E_i} = \dfrac{n_2 \cos\theta_1 - n_1 \cos\theta_2}{n_2 \cos\theta_1 + n_1 \cos\theta_2}$

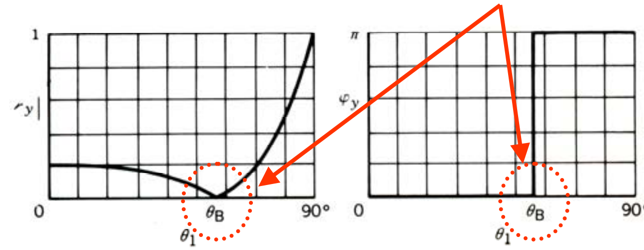
External Reflection of TE Wave (Low to High)



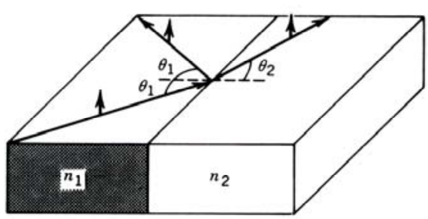
External Reflection of TM Wave (Low to High)



Brewster Angle: $\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right)$



Internal Reflection of TE Wave (High to Low)

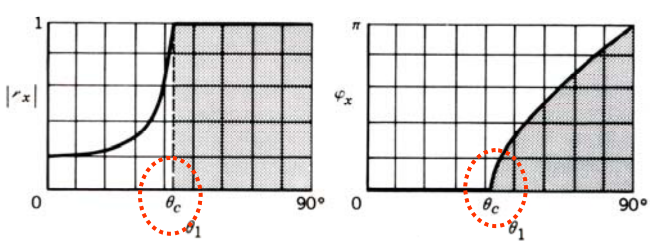


Snell's law:

$$\frac{n_1}{n_2} = \frac{\sin(\theta_2)}{\sin(\theta_1)}$$

Critical Angle:

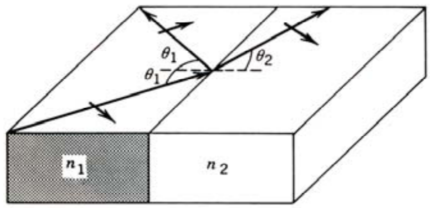
$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$



For $\theta_1 > \theta_c$

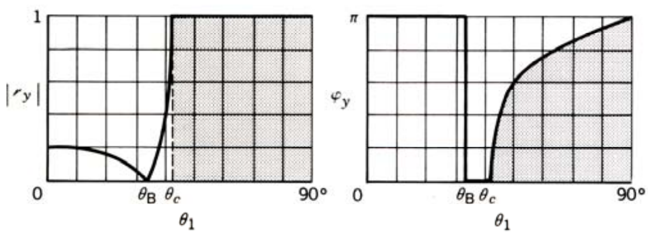
$$\tan\frac{\varphi}{2} = \frac{\sqrt{\sin^2\theta_1 - \sin^2\theta_c}}{\cos\theta_1}$$

Internal Reflection of TM Wave (High to Low)



Critical Angle:

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$



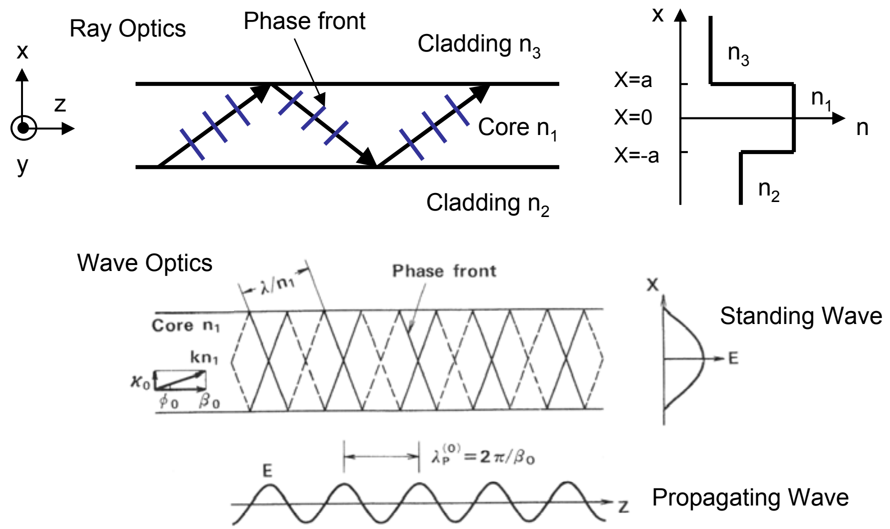
For $\theta_1 > \theta_c$

$$\tan\frac{\varphi}{2} = \frac{\sqrt{\sin^2\theta_1 - \sin^2\theta_c}}{\cos\theta_1 \sin^2\theta_c}$$

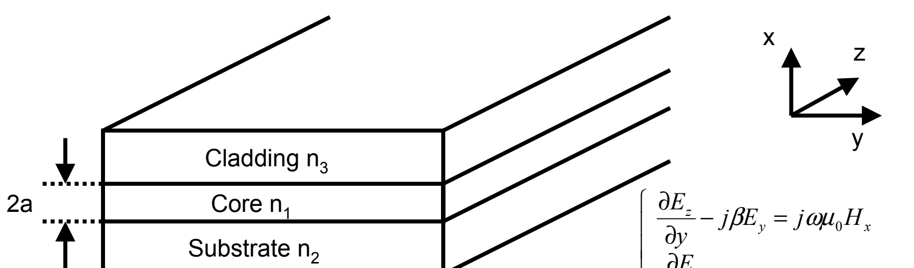
Summary of Reflection at Interface

- **The phase changes either by π or by 0 for external reflection.**
- **The phase has abrupt change across Brewster angle for TM wave.**
- **The Brewster angle is smaller than critical angle.**
- **The phase changes continuously for total internal reflection**

Slab Waveguide (1D Waveguide)



Derivation of Basic Equations



General Waveguide Mode Expression:

$$\begin{cases} \tilde{E} = E(x,y)\exp[j(\beta z - \omega t)] \\ \tilde{H} = H(x,y)\exp[j(\beta z - \omega t)] \end{cases}$$

Transverse Component — Longitudinal Component

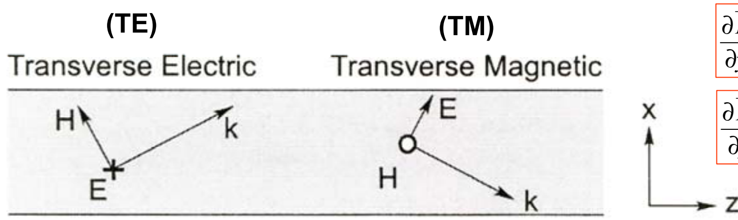
F. Law

$$\begin{cases} \dfrac{\partial E_z}{\partial y} - j\beta E_y = j\omega\mu_0 H_x \\ -\dfrac{\partial E_z}{\partial x} + j\beta E_x = j\omega\mu_0 H_y \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = j\omega\mu_0 H_z \end{cases}$$

A. Law

$$\begin{cases} \dfrac{\partial H_z}{\partial y} - j\beta H_y = -j\omega\varepsilon_0 n^2 E_x \\ -\dfrac{\partial H_z}{\partial x} + j\beta H_x = -j\omega\varepsilon_0 n^2 E_y \\ \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} = -j\omega\varepsilon_0 n^2 E_z \end{cases}$$

Polarization of Optical Fields in Slab (Planar) Waveguide



$$\frac{\partial \vec{E}}{\partial y} = 0$$

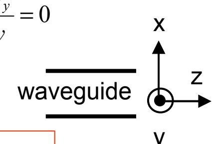
$$\frac{\partial \vec{H}}{\partial y} = 0$$

E,H is independent of y-axis

- **Transverse Electric (TE): $E_z = 0$, $H_z \neq 0$**
  ➔$E_x = H_y = 0$
- **Transverse Magnetic (TM): $H_z = 0$, $E_z \neq 0$**
  ➔$H_x = E_y = 0$
- **For slab waveguide, the optical modes can always be decomposed into TE and TM**

TE-Polarized Optical Modes in Slab Waveguide

E_z, E_x = 0 and E_y is y-direction independent; that is, $\frac{\partial E_y}{\partial y} = 0$

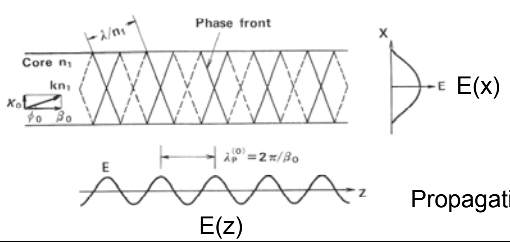


The expression of optical wave:

$$\vec{\mathbf{E}}(r,t) = \vec{\mathbf{E}}(r)\exp(-j\omega t) \longrightarrow \vec{\mathbf{E}}(r,t) = \mathbf{E}_y(x,z)\exp(-j\omega t)$$

The optical wave propagates in z-direction

$$\vec{\mathbf{E}}(r,t) = \mathbf{E}_y(x,z)\exp(-j\omega t) \longrightarrow \vec{\mathbf{E}}(r,t) = \mathbf{E}_y(x)\exp(j\beta z)\exp(-j\omega t) = \mathbf{E}_y(x)\exp(j\beta z - j\omega t)$$



TE-Polarized Optical Modes in Slab Waveguide

$$\nabla^2 \vec{\mathbf{E}}(r) + n(\omega)^2 k_0^{\ 2} \cdot \vec{\mathbf{E}}(r) = 0$$

$$\downarrow \quad \vec{\mathbf{E}}(r,t) = \mathbf{E}_y(x)\exp(j\beta z - j\omega t)$$

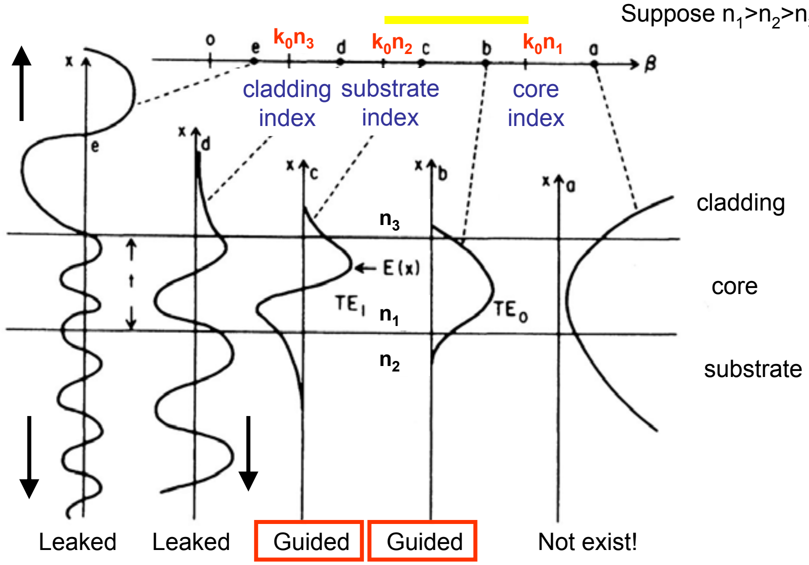
$$\begin{cases} \dfrac{\partial^2 \mathbf{E}_y}{\partial x^2} + (n^2 k_0^{\ 2} - \beta^2)\mathbf{E}_y = 0 \quad (1) \\ \mathbf{H}_x = -\dfrac{\beta}{\omega\mu_0}\mathbf{E}_y \\ \mathbf{H}_z = -\dfrac{j}{\omega\mu_0}\dfrac{d\mathbf{E}_y}{dx} \end{cases}$$

n could be in core, substrate or cladding

What is the solution of (1)?

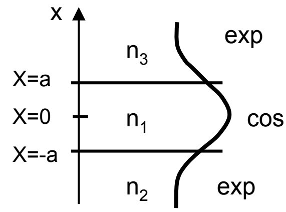
What is the β ?

Suppose $n_1 > n_2 > n_3$



TE-Polarized Optical Guided Modes in Slab Waveguide

$$E_y = \begin{cases} A\cos(h_1 a-\phi)\exp[-\gamma_3(x-a)] & (x>a) \\ A\cos(h_1 x-\phi) & (-a\le x\le a) \\ A\cos(h_1 a+\phi)\exp[\gamma_2(x+a)] & (x<-a) \end{cases}$$



where

$$\begin{cases} h_1 = \sqrt{k_0^2 n_1^2 - \beta^2} \\ \gamma_3 = \sqrt{\beta^2 - k_0^2 n_3^2} \\ \gamma_2 = \sqrt{\beta^2 - k_0^2 n_2^2} \end{cases}$$

Boundary Condition 1: E_y **is continuous at x = a and x = -a**

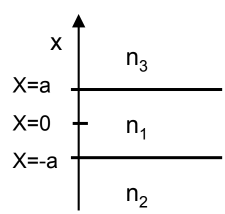
Trivial !

Boundary Condition 2: H_z **is continuous at x = a and x = -a**

$\mathbf{H}_z = -\dfrac{j}{\omega\mu_0}\dfrac{d\mathbf{E}_y}{dx}$ should be continuous at the boundary

TE-Polarized Optical Modes in Slab Waveguide

$$\frac{dE_y}{dx} = \begin{cases} -\gamma_3 A\cos(h_1 a-\phi)\exp[-\gamma_3(x-a)] & (x>a) \\ -h_1 A\sin(h_1 x-\phi) & (-a\le x\le a) \\ \gamma_2 A\cos(h_1 a+\phi)\exp[\gamma_2(x+a)] & (x<-a) \end{cases}$$



$$\left.\frac{dE_y}{dx}\right|_{x=a^-} = \left.\frac{dE_y}{dx}\right|_{x=a^+} \quad \textbf{and} \quad \left.\frac{dE_y}{dx}\right|_{x=-a^-} = \left.\frac{dE_y}{dx}\right|_{x=-a^+}$$

$$\begin{cases} h_1\sin(h_1 a+\phi) = \gamma_2\cos(h_1 a+\phi) \\ h_1\sin(h_1 a-\phi) = \gamma_3\cos(h_1 a-\phi) \end{cases} \longrightarrow \begin{cases} \tan(u+\phi) = \dfrac{w}{u} \\ \tan(u-\phi) = \dfrac{w'}{u} \end{cases} \quad \textbf{where} \quad \begin{cases} u = h_1 a \\ w = \gamma_2 a \\ w' = \gamma_3 a \end{cases}$$

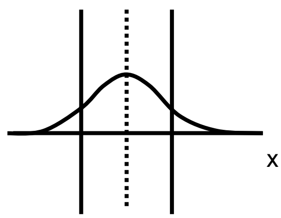
$$\begin{cases} u = \dfrac{m\pi}{2} + \dfrac{1}{2}\tan^{-1}(\dfrac{w}{u}) + \dfrac{1}{2}\tan^{-1}(\dfrac{w'}{u}) \\ \phi = \dfrac{m\pi}{2} + \dfrac{1}{2}\tan^{-1}(\dfrac{w}{u}) - \dfrac{1}{2}\tan^{-1}(\dfrac{w'}{u}) \end{cases} \qquad m = 0,1,2,\ldots$$

Symmetric and Asymmetric

Symmetric

$$\gamma_2 = \gamma_3 = \gamma$$
$$w = w^{'}$$

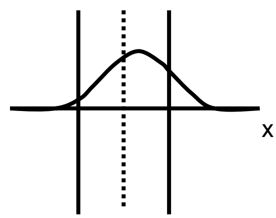
$$\begin{cases} u = \dfrac{m\pi}{2} + \tan^{-1}(\dfrac{w}{u}) \\ \phi = \dfrac{m\pi}{2} \end{cases}$$



Asymmetric

$$\gamma_2 \neq \gamma_3$$
$$w \neq w^{'}$$

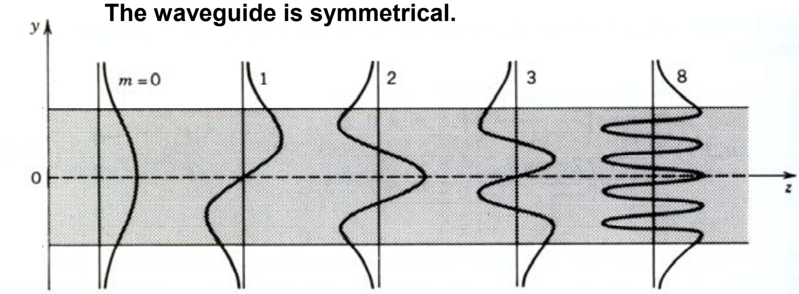
$$\begin{cases} u = \dfrac{m\pi}{2} + \dfrac{1}{2}\tan^{-1}(\dfrac{w}{u}) + \dfrac{1}{2}\tan^{-1}(\dfrac{w^{'}}{u}) \\ \phi = \dfrac{m\pi}{2} + \dfrac{1}{2}\tan^{-1}(\dfrac{w}{u}) - \dfrac{1}{2}\tan^{-1}(\dfrac{w^{'}}{u}) \end{cases}$$



Electric Fields of High Order Mode

$$E_y = \begin{cases} A\cos(h_1 a - \phi)\exp[-\gamma_3(x-a)] & (x > a) \\ A\cos(h_1 x - \phi) & (-a \le x \le a) \\ A\cos(h_1 a + \phi)\exp[\gamma_2(x+a)] & (x < -a) \end{cases}$$

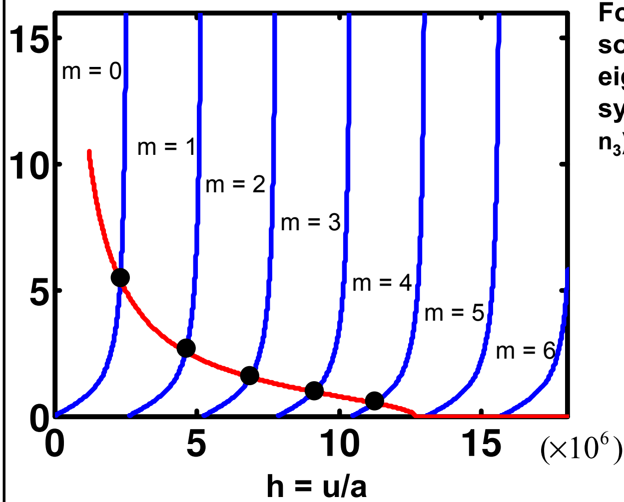
The waveguide is symmetrical.



m: even → cos(hx)

m: odd → sin(hx)

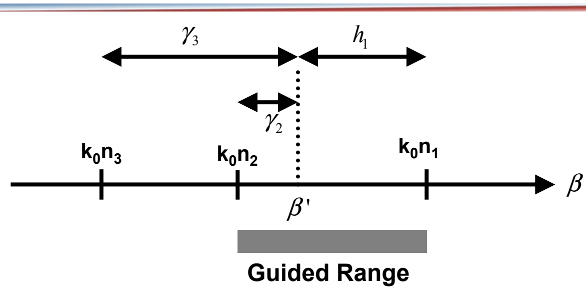
Mode Number



For example, how many solutions to satisfy the eigenvalue equation for a symmetric waveguide ($n_2 = n_3$)?

$$\tan(u + \frac{m\pi}{2}) = \frac{w}{u}$$

Cut-Off Condition



For a guided mode,

$$k_0 n_2 < \beta < k_0 n_1 \longrightarrow 0 < h_1 < \sqrt{k_0^2 (n_1^2 - n_2^2)}$$

We define $V \equiv a\sqrt{k_0^2 (n_1^2 - n_2^2)}$

$$u = h_1 a < a\sqrt{k_0^2 (n_1^2 - n_2^2)} = V$$

High-order modes have large u: $u(m+1) > u(m)$

Cut-Off Condition

What is the highest mode can be supported in the waveguide?

Recall $$u(m)=\frac{m\pi}{2}+\frac{1}{2}\tan^{-1}\left(\frac{w(m)}{u(m)}\right)+\frac{1}{2}\tan^{-1}\left(\frac{w(m)'}{u(m)}\right)$$

Suppose the highest mode No. is M, then u(M) is approximately given by

$u(M)\simeq V$ (close to cut-off)

Therefore,

$$\begin{cases} u(M)\approx a\sqrt{k_0^2(n_1^2-n_2^2)} \\ w(M)'\approx a\sqrt{k_0^2(n_2^2-n_3^2)} \\ w(M)=0 \end{cases}$$

$$u(M)=\frac{M\pi}{2}+\frac{1}{2}\tan^{-1}\left(\frac{\sqrt{(n_2^2-n_3^2)}}{\sqrt{(n_1^2-n_2^2)}}\right)$$

Cut-Off Condition

As we know, u(M) has to be less than V; that is,

$$u(M)=\frac{M\pi}{2}+\frac{1}{2}\tan^{-1}\left(\frac{\sqrt{(n_2^2-n_3^2)}}{\sqrt{(n_1^2-n_2^2)}}\right)<V$$

Therefore, for a waveguide, the accommodated mode no (M).

$$M_{TE}=\left\langle\frac{2V}{\pi}-\frac{1}{\pi}\tan^{-1}\left(\sqrt{\frac{n_2^2-n_3^2}{n_1^2-n_2^2}}\right)\right\rangle_{\text{int}}$$

For symmetric waveguides, there is always a TE mode (m=0) existing.

TM-Polarized Optical Modes in Slab Waveguide

Homework:

$$\begin{cases} \dfrac{\partial^2 \mathbf{E}_y}{\partial x^2} + (n^2 k_0{}^2 - \beta^2)\mathbf{E}_y = 0 \\ \mathbf{H}_x = -\dfrac{\beta}{\omega\mu_0}\mathbf{E}_y \\ \mathbf{H}_z = -\dfrac{j}{\omega\mu_0}\dfrac{d\mathbf{E}_y}{dx} \end{cases} \qquad \underset{H}{\mu_0} \overset{\text{Reciprocal}}{\longleftrightarrow} \underset{E}{\varepsilon_0 n^2} \qquad \begin{cases} \dfrac{d^2 \mathbf{H}_y}{dx^2} + (n^2 k_0{}^2 - \beta^2)\mathbf{H}_y = 0 \\ \mathbf{E}_x = \dfrac{\beta}{\omega\varepsilon_0 n^2}\mathbf{H}_y \\ \mathbf{E}_z = \dfrac{j}{\omega\varepsilon_0 n^2}\dfrac{d\mathbf{H}_y}{dx} \end{cases}$$

1. What is the cut-off condition for TM mode?
2. For a silicon slab with 0.25-um thickness (n=3.4, air-cladding on the two sides), what are the β_s of TM mode and TE mode (m=0) for wavelength from 1520 nm to 1560 nm?

Optical Power (Background)

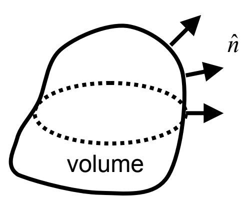
$\vec{S} \equiv \vec{E}\times\vec{H}$ **Poynting vector**

$$\nabla\cdot\vec{S} \equiv \nabla\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot\nabla\times\vec{E} - \vec{E}\cdot\nabla\times\vec{H}$$

$$\iiint_{Volume} \nabla\cdot\vec{A}\,dv = \oiint_{surface} \vec{A}\cdot\hat{n}\,ds$$

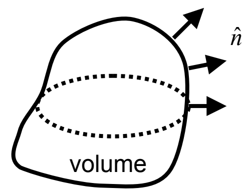
$$\iiint_{Volume} (\vec{H}\cdot\nabla\times\vec{E} - \vec{E}\cdot\nabla\times\vec{H})dv = \oiint_{surface} (\vec{E}\times\vec{H})\cdot\hat{n}\,ds$$

$$\iiint_{Volume} (\varepsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \mu\vec{H}\cdot\frac{\partial\vec{H}}{\partial t})dv = -\oiint_{surface} (\vec{E}\times\vec{H})\cdot\hat{n}\,ds$$



Optical Power (Background)

$$\iiint_{Volume} \left(\varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t}\right) dv = -\oiint_{surface} (\vec{E} \times \vec{H}) \cdot \hat{n}\, ds$$



$$\frac{\partial}{\partial t}\left(\frac{\varepsilon}{2}\left|\vec{E}\right|^2\right) \equiv \frac{\partial}{\partial t} W_e$$

Electric Stored Energy

$$\frac{\partial}{\partial t}\left(\frac{\varepsilon}{2}\left|\vec{H}\right|^2\right) \equiv \frac{\partial}{\partial t} W_h$$

Magnetic Stored Energy

$$\frac{\partial}{\partial t} \iiint_{Volume} (W_e + W_h)\, dv = -\oiint_{surface} \vec{S} \cdot \hat{n}\, ds$$

S represents the power flow density

Optical Power (Background)

The intensity of the power density can be calculated through the time-average of the poynting vector over one wave cycle:

$$\left\langle \vec{S} \right\rangle = \frac{1}{T} \int_0^T \vec{E} \times \vec{H}\, dt$$

For monochromatic wave

$$\vec{E} = \frac{1}{2}\left(\vec{\mathbf{E}} \exp(-j\omega t) + \vec{\mathbf{E}}^* \exp(j\omega t)\right)$$

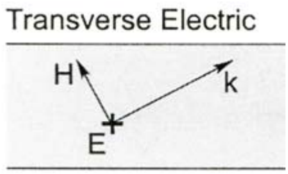
$$\vec{H} = \frac{1}{2}\left(\vec{\mathbf{H}} \exp(-j\omega t) + \vec{\mathbf{H}}^* \exp(j\omega t)\right)$$

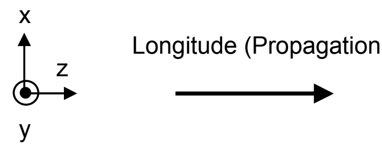
$$\left\langle \vec{S} \right\rangle = \frac{1}{2} \mathrm{Re}(\vec{\mathbf{E}} \times \vec{\mathbf{H}}^*)$$

$$P = \iint_s \left\langle \vec{S} \right\rangle \cdot \hat{u}_s\, ds = \iint_s \frac{1}{2} \mathrm{Re}(\vec{\mathbf{E}} \times \vec{\mathbf{H}}^*) \cdot \hat{u}_s\, ds$$

Optical Power of Slab Waveguide

For TE mode, the only non-zero component is E_y. And because wave is propagate in z-direction, we only consider E_y and H_x. Therefore, the power of TE mode is



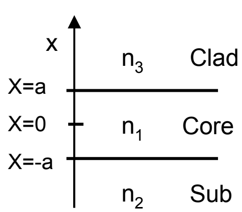


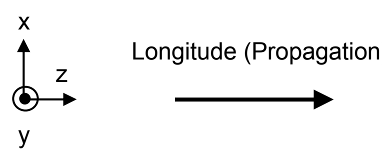
$$P = \iint_s \langle \vec{S} \rangle \cdot \hat{u}_s ds = \iint_s \frac{1}{2} \mathrm{Re}(\vec{\mathbf{E}} \times \vec{\mathbf{H}}^*) \cdot \hat{u}_s ds$$

$$P_{TE} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \mathrm{Re}(\mathbf{E}_y \cdot \mathbf{H}_x^*) dxdy = \frac{\beta}{2\omega\mu_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| E_y \right|^2 dxdy$$

Recall $E_y = \frac{1}{2}\left(\mathbf{E}_y + \mathbf{E}_y^*\right)$

Optical Power of Slab Waveguide





For slab waveguide (TE polarization),

Suppose the field amplitude is A

$$P_{TE} = \frac{\beta}{2\omega\mu_0} \int_0^1 dy \int_{-\infty}^{\infty} \left| E_y \right|^2 dx$$

$$P_{Core} = \frac{\beta a A^2}{2\omega\mu_0} \left\{ 1 + \frac{\sin^2(u+\phi)}{2w} + \frac{\sin^2(u-\phi)}{2w'} \right\}$$

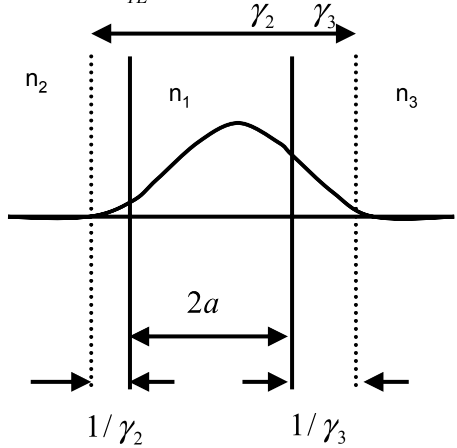
$$P_{Sub} = \frac{\beta a A^2}{2\omega\mu_0} \left\{ \frac{\cos^2(u+\phi)}{2w} \right\}$$

$$P_{Clad} = \frac{\beta a A^2}{2\omega\mu_0} \left\{ \frac{\cos^2(u-\phi)}{2w'} \right\}$$

$$P = P_{Core} + P_{Sub} + P_{clad} = \frac{\beta a A^2}{2\omega\mu_0} \left\{ 1 + \frac{1}{2w} + \frac{1}{2w'} \right\}$$

Effective Waveguide Thickness

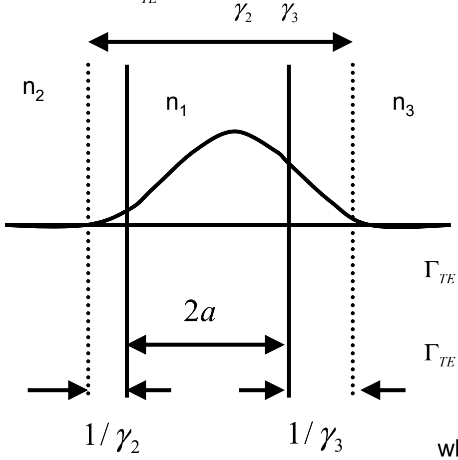
$$d_{TE} = 2a + \frac{1}{\gamma_2} + \frac{1}{\gamma_3}$$



samller γ represents the optical field penetrates farther

Mode Confinement

$$d_{TE} = 2a + \frac{1}{\gamma_2} + \frac{1}{\gamma_3}$$



$$\Gamma_{TE} = \frac{\int_{-d/2}^{d/2} |E_y(x)|^2 \, dx}{\int_{-\infty}^{\infty} |E_y(x)|^2 \, dx}$$

$$\Gamma_{TM} = \frac{\int_{-d/2}^{d/2} n_1^{-2} |H_y(x)|^2 \, dx}{\int_{-\infty}^{\infty} n(x)^{-2} |H_y(x)|^2 \, dx}$$

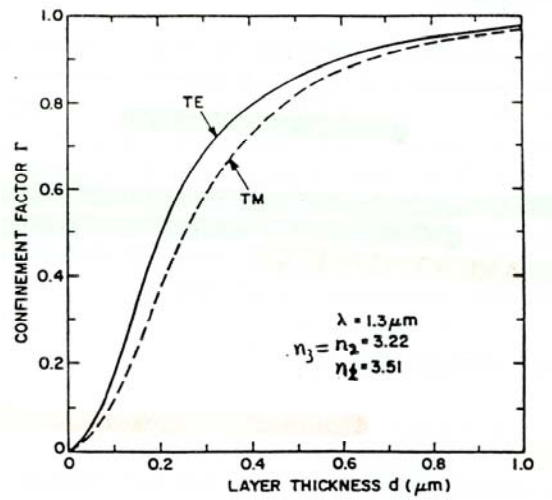
$$\Gamma_{TE} = \frac{1}{d_{TE}} \left(2a + \frac{1}{\gamma_2} \cdot \frac{1}{1 + h_1^2/\gamma_2^2} + \frac{1}{\gamma_3} \cdot \frac{1}{1 + h_1^2/\gamma_3^2}\right)$$

$$\Gamma_{TE} = \frac{1}{d_{TM}} \left(2a + \frac{1}{\gamma_2 q_2} \cdot \frac{1}{1 + h_1^2/\gamma_2^2} + \frac{1}{\gamma_3 q_3} \cdot \frac{1}{1 + h_1^2/\gamma_3^2}\right)$$

where $q_2 = \frac{\beta^2}{k_1^2} + \frac{\beta^2}{k_2^2} - 1 \qquad q_3 = \frac{\beta^2}{k_1^2} + \frac{\beta^2}{k_3^2} - 1$

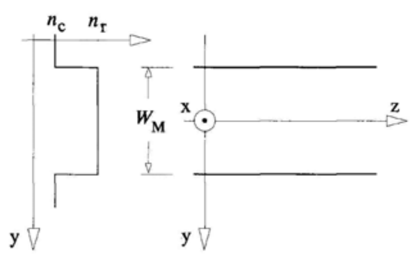
$$d_{TM} = 2a + \frac{1}{\gamma_2 q_2} + \frac{1}{\gamma_3 q_3}$$

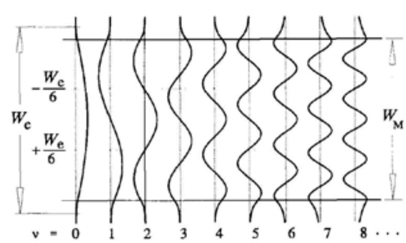
Mode Confinement



- **Low order modes have better mode confinement than high order modes**
- **TE modes have better mode confinement than TM modes**

Approximated Solutions of Waveguide Modes





$h_{y\nu}^2 + \beta_\nu^2 = k_0^2 n_r^2$ **v = 0,1,2,…**

where $k_0 = \dfrac{2\pi}{\lambda_0}$ **and** $h_{y\nu} = \dfrac{(\nu+1)\pi}{W_{e\nu}}$ ← Effective width

Suppose the waveguide is well-confined

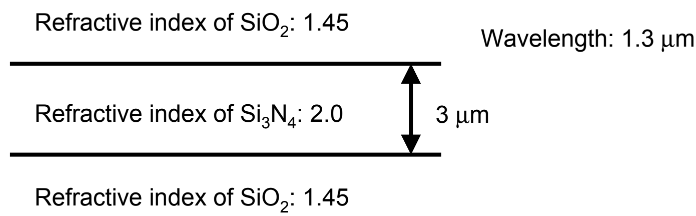
$$W_{e0} = W_M + \left(\frac{\lambda_0}{\pi}\right)\left(\frac{n_c}{n_r}\right)^{2\sigma}(n_r^2 - n_c^2)^{-\frac{1}{2}}$$

$\sigma = 1$ **for TM**, 0 **for TE**

Symmetric Waveguide

$\beta_0 \simeq k_0 n_r$ (Fundametial Mode)

Example



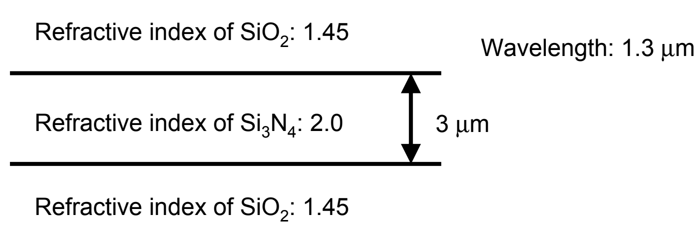
1. How many TE modes?

$$V = \frac{2\pi}{\lambda}\cdot a\cdot\sqrt{\left(n_r^{\,2}-n_c^{\,2}\right)} = \frac{2\pi}{1.3}\cdot\frac{3}{2}\cdot\sqrt{\left(2^2-1.45^2\right)} = 9.986$$

$$M_{TE} = \left\langle \frac{2V}{\pi} - \frac{1}{\pi}\tan^{-1}\left(\sqrt{\frac{n_2^{\,2}-n_3^{\,2}}{n_1^{\,2}-n_2^{\,2}}}\right)\right\rangle_{\text{int}} = \left\langle\frac{2V}{\pi}\right\rangle_{\text{int}} = \left\langle\frac{2\cdot 9.986}{\pi}\right\rangle_{\text{int}} = 6 \rightarrow \text{7 TE modes}$$

for symmetric waveguide

Example



2. What is the propagation constant β of the fundamental mode (TE)?

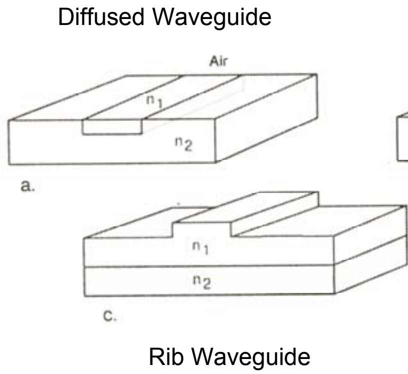
$$W_{e0} = W_M + \left(\frac{\lambda_0}{\pi}\right)\left(\frac{n_c}{n_r}\right)^{2\sigma}\left(n_r^{\,2}-n_c^{\,2}\right)^{-\frac{1}{2}} \longrightarrow W_{e0} = W_M + \left(\frac{\lambda_0}{\pi}\right)\left(n_r^{\,2}-n_c^{\,2}\right)^{-\frac{1}{2}} \quad \text{(TE)}$$

$$W_{e0} = 3 + \left(\frac{1.3}{\pi}\right)\left(2^2-1.45^2\right)^{-\frac{1}{2}} = 3.3(\mu m) \qquad h_0 = \frac{(0+1)\pi}{W_{e0}} = \frac{\pi}{3.3}$$

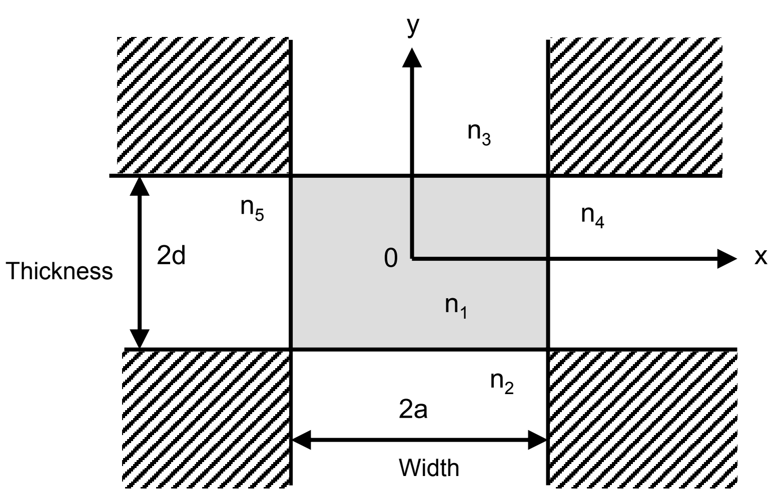
$$\beta_0 = \sqrt{k_0^{\,2}n_r^{\,2}-h_0^{\,2}} = \sqrt{\left(\frac{2\pi}{1.3}\cdot 2\right)^2 - \left(\frac{\pi}{3.3}\right)^2} = 9.6194(\mu m^{-1})$$

actual value: (9.6664)

Rectangular Waveguide (2D Waveguide)



Index Profile of A Channel Waveguide



$n_1 > n_2, n_3, n_4, n_5$

Mode Expression

Magnetic field intensity: $\mathbf{H}_x(x,y)=\mathbf{X}(x)\mathbf{Y}(y)\sim\exp(-jh_x x+\phi_x)\exp(-jh_y y+\psi_y)$

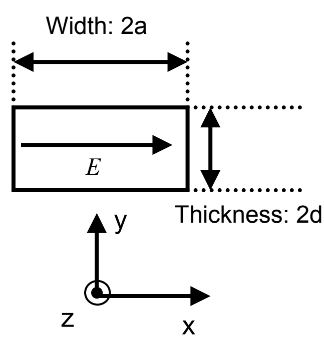
		X = -a		X = a	
	$\exp[\gamma_x(x+a)]$ $\exp[-\gamma_y(y-d)]$		$\cos(h_x x+\phi_x)$ $\exp[-\gamma_y(y-d)]$		$\exp[-\gamma_x(x-a)]$ $\exp[-\gamma_y(y-d)]$
Y = d					
	$\exp[\gamma_x(x+a)]$ $\cos(h_y y+\psi_y)$		$\cos(h_x x+\phi_x)$ $\cos(h_y y+\psi_y)$		$\exp[-\gamma_x(x-a)]$ $\cos(h_y y+\psi_y)$
Y = -d					
	$\exp[\gamma_x(x+a)]$ $\exp[\gamma_y(y+d)]$		$\cos(h_x x+\phi_x)$ $\exp[\gamma_y(y+d)]$		$\exp[-\gamma_x(x-a)]$ $\exp[\gamma_y(y+d)]$

TE Polarization (TE-Like)

H_x = 0, E_x and H_y is predominant

$$\frac{d^2\mathbf{H}_y}{dx^2}+\frac{\partial^2\mathbf{H}_y}{\partial y^2}+(n^2k_0{}^2-\beta^2)\mathbf{H}_y=0$$

$$\begin{cases}\mathbf{H}_x=0\\ \mathbf{E}_x=\dfrac{\omega\mu_0}{\beta}\mathbf{H}_y+\dfrac{1}{\omega\varepsilon_0 n^2\beta}\dfrac{\partial^2\mathbf{H}_y}{\partial x^2}\\ \mathbf{E}_y=-\dfrac{1}{\omega\varepsilon_0 n^2}\dfrac{\partial^2\mathbf{H}_y}{\partial x\partial y}\\ \mathbf{E}_z=\dfrac{j}{\omega\varepsilon_0 n^2}\dfrac{\partial\mathbf{H}_y}{\partial x}\\ \mathbf{H}_z=\dfrac{j}{\beta}\dfrac{\partial\mathbf{H}_y}{\partial y}\end{cases}$$



Suppose width > thickness

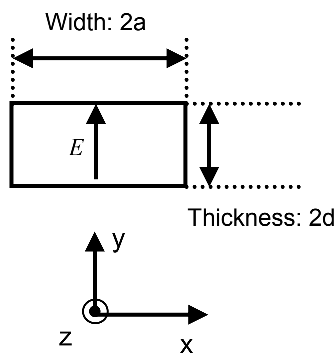
This modelling was proposed by Marcatili

TM Polarization (TM-like)

$H_y = 0$, E_y and H_x is predominant

$$\frac{d^2\mathbf{H}_x}{dx^2}+\frac{\partial^2\mathbf{H}_x}{\partial y^2}+(n^2k_0{}^2-\beta^2)\mathbf{H}_x=0$$

$$\begin{cases}\mathbf{H}_y=0\\ \mathbf{E}_y=-\dfrac{\omega\mu_0}{\beta}\mathbf{H}_x-\dfrac{1}{\omega\varepsilon_0 n^2\beta}\dfrac{\partial^2\mathbf{H}_x}{\partial y^2}\\ \mathbf{E}_x=\dfrac{1}{\omega\varepsilon_0 n^2}\dfrac{\partial^2\mathbf{H}_x}{\partial x\partial y}\\ \mathbf{E}_z=\dfrac{-j}{\omega\varepsilon_0 n^2}\dfrac{\partial\mathbf{H}_x}{\partial y}\\ \mathbf{H}_z=\dfrac{j}{\beta}\dfrac{\partial\mathbf{H}_x}{\partial x}\end{cases}$$

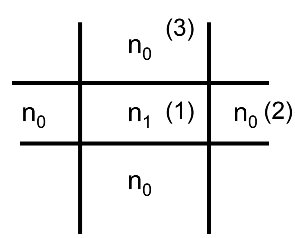


Suppose width > thickness

TE-Polarized Optical Modes in Channel Waveguide (Marcatili's Method)

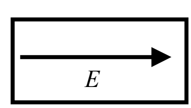
For simplicity, we assume the rectangular waveguide is symmetrical cladding

$$H_y=\begin{cases}A\cos(h_x x-\phi)\cos(h_y y-\psi) & (1)\\ A\cos(h_x a-\phi)\exp[-\gamma_x(x-a)]\cos(h_y y-\psi) & (2)\\ A\cos(h_x x-\phi)\exp[-\gamma_y(y-d)]\cos(h_y d-\psi) & (3)\end{cases}$$



Where

$$\begin{cases}-h_x{}^2-h_y{}^2+k_0{}^2n_1{}^2-\beta^2=0 & (1)\\ \gamma_x{}^2-h_y{}^2+k_0{}^2n_0{}^2-\beta^2=0 & (2)\\ -h_x{}^2+\gamma_y{}^2+k_0{}^2n_0{}^2-\beta^2=0 & (3)\end{cases}$$

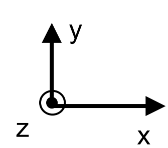


And

$$\begin{cases}\phi=(p-1)\dfrac{\pi}{2} & p = 1,2,\ldots\\ \psi=(q-1)\dfrac{\pi}{2} & q = 1,2,\ldots\end{cases}$$

↖ from 1 instead of 0

TE_{pq}



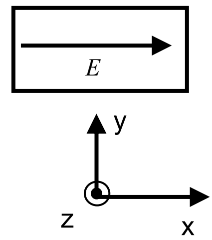
TE-Polarized Optical Modes in Channel Waveguide

Boundary Condition 1: E_z **is continuous at x = a, -a**

$$\mathbf{E}_z \propto \frac{1}{n^2}\frac{\partial \mathbf{H}_y}{\partial x}$$

Boundary Condition 2: H_z **is continuous at y = d, -d**

$$\mathbf{H}_z \propto \frac{\partial \mathbf{H}_y}{\partial y}$$



Given a p and q

$$\begin{cases} h_x a = (p-1)\dfrac{\pi}{2} + \tan^{-1}\left(\dfrac{n_1^2 \gamma_x}{n_0^2 h_x}\right) & \text{B.C. 1 (TM eigenvalue eq. in slab waveguide)} \\ h_y d = (q-1)\dfrac{\pi}{2} + \tan^{-1}\left(\dfrac{\gamma_y}{h_y}\right) & \text{B.C. 2 (TE eigenvalue eq. in slab waveguide)} \end{cases}$$

And

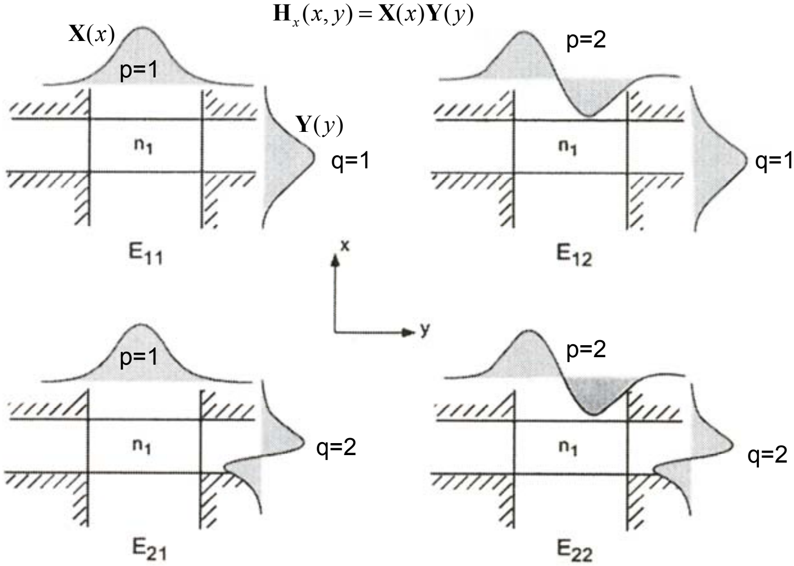
$$\begin{cases} \beta^2 = k_0^2 n_1^2 - (h_x^2 + h_y^2) \\ \gamma_x^2 = k_0^2 (n_1^2 - n_0^2) - h_x^2 \\ \gamma_y^2 = k_0^2 (n_1^2 - n_0^2) - h_y^2 \end{cases}$$

Five unknowns:

$$\beta, h_x, h_y, \gamma_x, \gamma_y$$

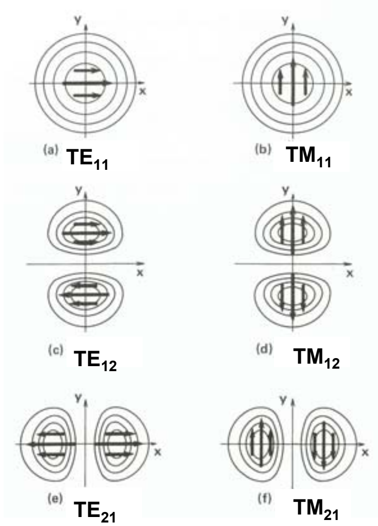
Mode Profile

$$\mathbf{H}_x(x,y) = \mathbf{X}(x)\mathbf{Y}(y)$$

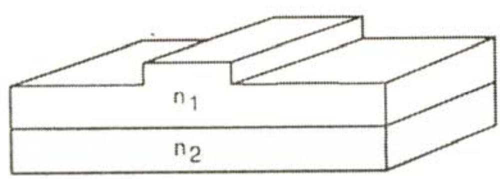


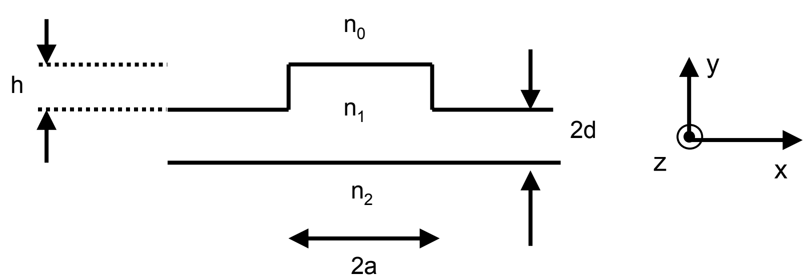
Mode Profile (Consider Polarization)

Electric field



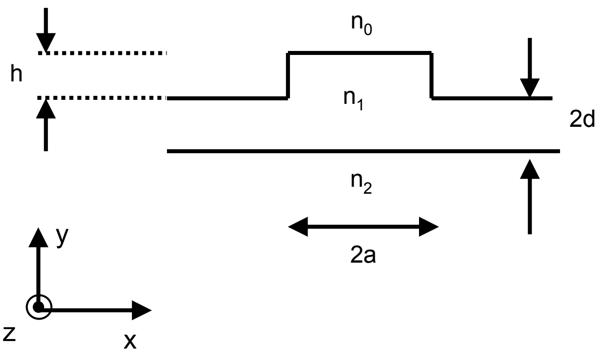
Rib Waveguide





It is difficult to be analyzed by Marcatili method!

Effective Index Methods



We suppose the polarization is in x-direction (TE). The wave equation for the TE mode is

$$\frac{\partial^2 \mathbf{H}_y}{\partial x^2}+\frac{\partial^2 \mathbf{H}_y}{\partial y^2}+[k_0{}^2 n^2(x,y)-\beta^2]\mathbf{H}_y=0$$

Effective Index Methods

Suppose the electromagnetic fields can be expressed with the separation of variables

$$\mathbf{H}_y(x,y)=\mathbf{X}(x)\mathbf{Y}(y)$$

$$\frac{\partial^2 \mathbf{H}_y}{\partial x^2}+\frac{\partial^2 \mathbf{H}_y}{\partial y^2}+[k_0{}^2 n^2(x,y)-\beta^2]\mathbf{H}_y=0$$

↓ Divided by XY

$$\frac{1}{\mathbf{X}}\frac{d^2\mathbf{X}}{dx^2}+\frac{1}{\mathbf{Y}}\frac{d^2\mathbf{Y}}{dy^2}+[k_0{}^2 n^2(x,y)-\beta^2]=0$$

↓ $N_{eff}(x)$: effective index distribution

$$\begin{cases}\dfrac{1}{\mathbf{Y}}\dfrac{d^2\mathbf{Y}}{dy^2}+[k_0{}^2 n^2(x,y)-k_0 n_{eff}{}^2(x)]=0\\[2ex]\dfrac{1}{\mathbf{X}}\dfrac{d^2\mathbf{X}}{dx^2}+[k_0{}^2 n_{eff}{}^2(x,y)-\beta^2]=0\end{cases}$$

Effective Index Methods

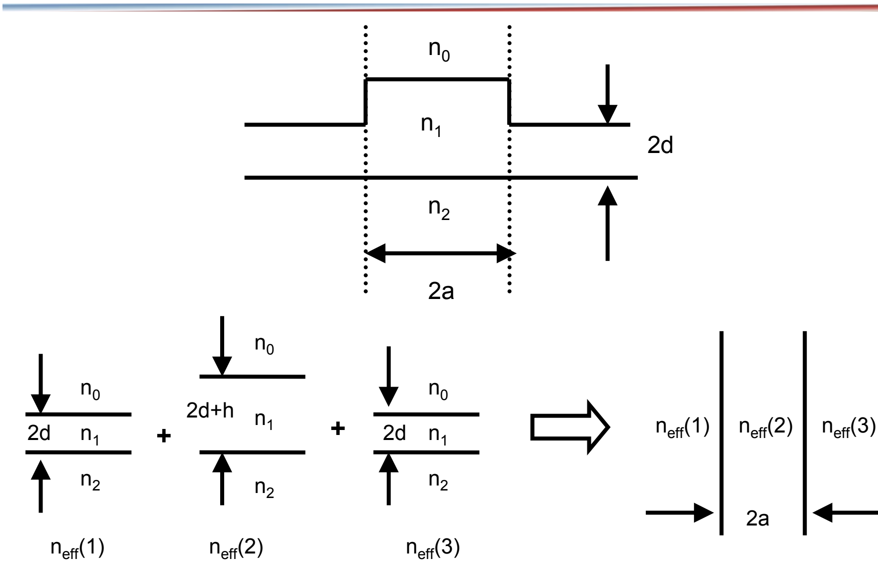
$$\begin{cases} \frac{1}{\mathbf{Y}}\frac{d^2\mathbf{Y}}{dy^2} + [k_0^2 n^2(x,y) - k_0^2 n_{eff}^2(x)] = 0 \\ \frac{1}{\mathbf{X}}\frac{d^2\mathbf{X}}{dx^2} + [k_0^2 n_{eff}^2(x,y) - \beta^2] = 0 \end{cases} \rightarrow \begin{cases} \frac{d^2\mathbf{Y}}{dy^2} + [k_0^2 n^2(x,y) - k_0^2 n_{eff}^2(x)]\mathbf{Y} = 0 \\ \frac{d^2\mathbf{X}}{dx^2} + [k_0^2 n_{eff}^2(x,y) - \beta^2]\mathbf{X} = 0 \end{cases}$$

Procedure of solving problems

- **Divide the waveguide into several sections (horizontally or vertically).**
- **Consider either Y component or X component only.**
- **Calculate the propagation constant in each section.**
- **Calculate the effective indices** $n_{eff} = \frac{\beta}{k_0}$
- **Consider the other component based on the effective indices**

2D rectangular waveguide problem → Several slab waveguide problems

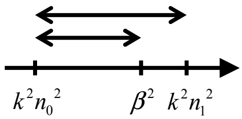
Effective Index Methods

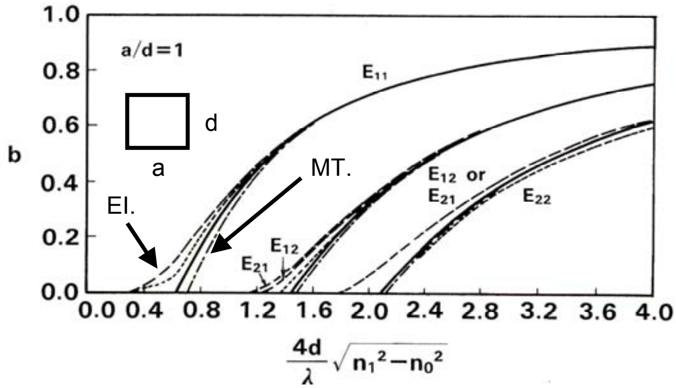


Comparison of Effective Index Method and Marcatili's Method

Normalized Index

$$b = \frac{\beta^2 - k^2 n_0^2}{k^2 n_1^2 - k^2 n_0^2}$$





- **For same waveguide, the propagation constant calculated from effective index method is larger than that of Marcatili's Method.**
- **The accurate propagation constant is between these two values.**