Background of Wave Optics

Class: Integrated Photonic Devices
Time: Fri. 8:00am ~ 11:00am.
Classroom: 資電206
Lecturer: Prof. 李明昌(Ming-Chang Lee)

Fundamentals of Electromagnetic Waves

Electrical field	$\vec{E}(r,t)$	V/m
Electrical displacement	$\vec{D}(r,t)$	C/m^2
Magnetic field	$\vec{H}(r,t)$	A/m
Magnetic induction	$\vec{B}(r,t)$	webers/m^2

space time

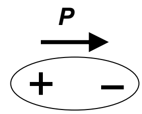
- **E and B are fundamental fields**
- **D and H are derived from the response of the medium**
- **How many variables in Electromagnetic Wave?**

Polarization and Magnetization

The relations between D,E and B,H

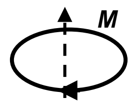
$$\begin{cases} \vec{D}(r,t) = \varepsilon_0 \vec{E}(r,t) + \vec{P}(r,t) \\ \vec{B}(r,t) = \mu_0 \vec{H}(r,t) + \mu_0 \vec{M}(r,t) \end{cases}$$

$\vec{\mathbf{P}}$(r,t): polarization (electric polarization)



Dipole

$\vec{\mathbf{M}}$(r,t): magnetization (magnetic polarization)



Current Loop

and

$$\begin{cases} \varepsilon_0 = \dfrac{1}{36\pi} \times 10^{-9} \text{ F/m} & \text{Electric permittivity of free space} \\ \mu_0 = 4\pi \times 10^{-7} \text{ H/m} & \text{Magnetic permeability of free space} \end{cases}$$

Basic Vector Operators

The Curl of A in Cartesian coordinates

$$\nabla \times A = \hat{e}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{e}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{e}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$= \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Vector → Vector

The Divergence of A in Cartesian coordinates

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_x}{\partial z}$$

Vector → Scalar

General Maxwell's Equations

Maxwell Equation:

$$\begin{cases} \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \text{Faraday's Law} \\ \nabla \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} & \text{Ampere's Law} \\ \nabla \cdot \vec{D} = \rho & \text{Coulomb's Law} \\ \nabla \cdot \vec{B} = 0 & \text{Absence of free magnetic monopole} \end{cases}$$

Plus:

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \text{Conservation of charges}$$

$\rho(r,t)$: Charge density (C/m^3)

$\vec{J}(r,t)$: Current density (A/m^2)

Maxwell's Equations in Dielectric Medium

In a medium free of source, J = 0 and ρ = 0. Then, Maxwell's equations are

$$\begin{cases} \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & (1) \\ \nabla \times \vec{H} = \dfrac{\partial \vec{D}}{\partial t} & (2) \\ \nabla \cdot \vec{D} = 0 & (3) \\ \nabla \cdot \vec{B} = 0 & (4) \end{cases}$$

At optical frequency,

$$\vec{B}(r,t) = \mu_0 \vec{H}(r,t) + \mu_0 \vec{M}(r,t) = \mu_0 \vec{H}(r,t)$$

Magnetization is not induced by optical fields but it can be induced by external magnetic sources such as magnets

Wave Equation

(1) and (2) are strongly coupled. To decouple the two curl eqs.

$$\nabla\times\nabla\times\vec{E}=\nabla\times-\frac{\partial\vec{B}}{\partial t}$$

$\vec{B}(r,t)=\mu_0\vec{H}(r,t)$

$$\nabla\times\nabla\times\vec{E}=\nabla\times-\frac{\partial\mu_0\vec{H}}{\partial t}=-\mu_0\left(\nabla\times\frac{\partial\vec{H}}{\partial t}\right)$$

$\vec{H}$ is continuous

$$\nabla\times\nabla\times\vec{E}=-\mu_0\frac{\partial}{\partial t}\left(\nabla\times\vec{H}\right)$$

(2)

$$\nabla\times\nabla\times\vec{E}=-\mu_0\frac{\partial}{\partial t}\frac{\partial\vec{D}}{\partial t}=-\mu_0\frac{\partial^2\vec{D}}{\partial t^2}$$

Wave Equation

$$\nabla\times\nabla\times\vec{E}=-\mu_0\frac{\partial}{\partial t}\frac{\partial\vec{D}}{\partial t}=-\mu_0\frac{\partial^2\vec{D}}{\partial t^2}$$

$\vec{D}(r,t)=\varepsilon_0\vec{E}(r,t)+\vec{P}(r,t)$

$$\nabla\times\nabla\times\vec{E}+\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}=-\mu_0\frac{\partial^2\vec{P}}{\partial t^2}$$

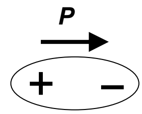
Wave Equation

Where $c=\frac{1}{\sqrt{\varepsilon_0\mu_0}}\sim 3\times10^8\,\mathbf{m/s}$

Electric Polarization

For linear medium,

$$\vec{P}(r,t) = \varepsilon_0 \int_{-\infty}^{\infty} dr' \int_{-\infty}^{t} dt' \chi(r-r', t-t') \cdot \vec{E}(r', t')$$



Dipole

$$\boxed{\vec{D}(r,t) = \varepsilon_0 \vec{E}(r,t) + \vec{P}(r,t)}$$

$$\vec{D}(r,t) = \varepsilon_0 \vec{E}(r,t) + \varepsilon_0 \int_{-\infty}^{\infty} dr' \int_{-\infty}^{t} dt' \chi(r-r', t-t') \cdot \vec{E}(r', t')$$

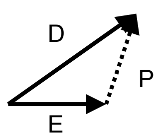
$$\equiv \int_{-\infty}^{\infty} dr' \int_{-\infty}^{t} dt' \varepsilon(r-r', t-t') \cdot \vec{E}(r', t')$$

$\varepsilon = \varepsilon_0 (I + \chi)$ is the electric permittivity tensor

χ is the electric susceptibility tensor

Electric Polarization

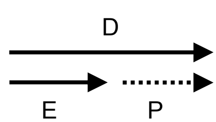
Anisotropy



$$\varepsilon = \varepsilon_0 (I + \chi)$$

permittivity tensor

Isotropy



$$\varepsilon = \varepsilon_0 (1 + \chi)$$

permittivity scalar

Complex Expression of Optical Field

- **Optical fields are harmonic fields which vary with time sinusoidally.**
- **It is convenient to define the harmonic fields as a complex fields**

For example,

$$\cos(\omega t) = \frac{1}{2}\exp(j\omega t) + \frac{1}{2}\exp(-j\omega t) = \frac{1}{2}\exp(j\omega t) + \frac{1}{2}\exp^*(j\omega t)$$

$$\begin{cases} \vec{E}(r,t) \equiv \frac{1}{2}\left(\vec{\mathrm{E}}(r,t) + \vec{\mathrm{E}}^*(r,t)\right) = \frac{1}{2}\left(\vec{\mathrm{E}}(r,t) + c.c\right) = \mathrm{Re}(\vec{\mathrm{E}}(r,t)) \\ \vec{P}(r,t) \equiv \frac{1}{2}\left(\vec{\mathrm{P}}(r,t) + \vec{\mathrm{P}}^*(r,t)\right) = \frac{1}{2}\left(\vec{\mathrm{P}}(r,t) + c.c\right) = \mathrm{Re}(\vec{\mathrm{P}}(r,t)) \\ \vec{D}(r,t) \equiv \frac{1}{2}\left(\vec{\mathrm{D}}(r,t) + \vec{\mathrm{D}}^*(r,t)\right) = \frac{1}{2}\left(\vec{\mathrm{D}}(r,t) + c.c\right) = \mathrm{Re}(\vec{\mathrm{D}}(r,t)) \end{cases}$$

$$\vec{\mathbf{D}}(r,t) = \int_{-\infty}^{\infty} dr' \int_{-\infty}^{t} dt' \varepsilon(r-r', t-t') \cdot \vec{\mathbf{E}}(r', t')$$

- **Wave equation is hold for complex expression**

Wave Propagation in Linear, Isotropic and Homogeneous Medium

General monochromatic wave: $\vec{\mathbf{E}}(r,t) = \vec{\mathbf{E}}(r)\exp(-j\omega t)$

For monochromatic waves in isotropic, homogeneous and space-nondispersive medium,

$$\begin{aligned} \vec{\mathbf{D}}(r,t) &= \int_{-\infty}^{\infty} dr' \int_{-\infty}^{t} dt' \delta(r-r')\varepsilon(t-t') \cdot \vec{\mathbf{E}}(r')\exp(-j\omega t') \\ &= \int_{-\infty}^{\infty} \vec{\mathbf{E}}(r')\delta(r-r')dr' \cdot \exp(-j\omega t) \underbrace{\int_{-\infty}^{t} \varepsilon(t-t')\exp[j\omega(t-t')]dt'}_{\text{Fourier Transform}} \\ &= \boldsymbol{\varepsilon}(\omega)\vec{\mathbf{E}}(r)\exp(-j\omega t) \\ &= \boldsymbol{\varepsilon}(\omega)\vec{\mathbf{E}}(r,t) \end{aligned}$$

$$\nabla \times \nabla \times \vec{\mathbf{E}} + \frac{1}{C^2}\frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = -\mu_0 \frac{\partial^2 \vec{\mathbf{P}}}{\partial t^2}$$

$$\Downarrow$$

$$\nabla \times \nabla \times \vec{\mathbf{E}} + \mu_0 \varepsilon(\omega) \cdot \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} = 0$$

Wave Propagation in Linear, Isotropic and Homogeneous Medium

$$\nabla\times\nabla\times\vec{\mathbf{E}} = \nabla(\nabla\cdot\vec{\mathbf{E}}) - \nabla^2\vec{\mathbf{E}}$$

(1) (2)

(1) ⟶ $\nabla\cdot\vec{\mathbf{E}} = -\vec{\mathbf{E}}\cdot\frac{\nabla\varepsilon}{\varepsilon}\left(\nabla\cdot\mathbf{D}=0\right)$

$= 0\,(\nabla\varepsilon = 0)$

(2) ⟶ $\nabla^2\vec{\mathbf{E}} = \nabla^2\mathbf{E}_x\hat{x} + \nabla^2\mathbf{E}_y\hat{y} + \nabla^2\mathbf{E}_z\hat{z}$

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla\times\nabla\times\vec{\mathbf{E}} + \mu_0\varepsilon(\omega)\cdot\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} = 0$$

↓

$n \equiv \sqrt{\frac{\varepsilon}{\varepsilon_0}}$ Refractive index

$$\nabla^2\vec{\mathbf{E}} - \mu_0\varepsilon(\omega)\cdot\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} = 0 \longrightarrow \nabla^2\vec{\mathbf{E}} - \frac{n(\omega)^2}{c^2}\cdot\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} = 0$$

Wave Propagation in Linear, Isotropic and Homogeneous Medium

$$\nabla^2\vec{\mathbf{E}}(r,t) - \frac{n(\omega)^2}{c^2}\cdot\frac{\partial^2\vec{\mathbf{E}}(r,t)}{\partial t^2} = 0$$

↓ $\vec{\mathbf{E}}(r,t) = \vec{\mathbf{E}}(r)\exp(-j\omega_0 t)$

$$\nabla^2\vec{\mathbf{E}}(r) + \frac{n(\omega_0)^2}{c^2}\omega_0^{\ 2}\cdot\vec{\mathbf{E}}(r) = 0$$

↓ $k_0 \equiv \frac{\omega_0}{c} = \frac{2\pi}{\lambda_0}$

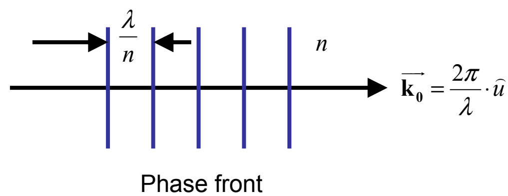
$$\boxed{\nabla^2\vec{\mathbf{E}}(r) + n(\omega_0)^2 k_0^{\ 2}\cdot\vec{\mathbf{E}}(r) = 0}$$ (Helmholtz equations)

We only consider the spatial terms

Plane Wave in Isotropic and Homogeneous Medium

Solution of plane wave:

$$\vec{\mathbf{E}}(r,t) \equiv \underbrace{\vec{\mathbf{E}_0}\exp(jn\vec{k_0}\cdot\vec{r})}_{\vec{\mathbf{E}}(r)}\cdot\exp(-j\omega t) = \vec{\mathbf{E}_0}\exp(jn\vec{k_0}\cdot\vec{r} - j\omega t)$$



- **If the constant polarization (E_0) is real vector, the wave is linearly polarized**
- **Otherwise, the wave is circularly polarized or elliptically polarized**

Summary of Wave Equation in Optical Domain

Maxwell's Equations

Dielectric Medium

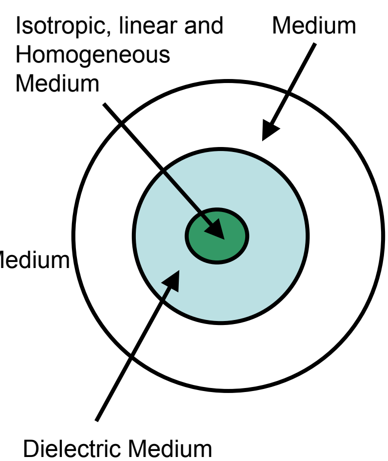
$$\nabla\times\nabla\times\vec{E} + \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = -\mu_0\frac{\partial^2\vec{P}}{\partial t^2}$$

Isotropic, linear and Homogeneous Medium

$$\nabla\times\nabla\times\vec{\mathbf{E}} + \mu_0\varepsilon(\omega)\cdot\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2} = 0$$

Monochromatic Waves

$$\nabla^2\vec{\mathbf{E}}(r) + n(\omega)^2 k_0{}^2\cdot\vec{\mathbf{E}}(r) = 0$$



Polarization of Light

Consider a monochromatic plane wave propagating in z direction

$$\vec{E} = \overrightarrow{E_0} \exp(jkz)$$

where E_0 is constant and lies on the x-y plane. There fore E_0 can be represented by the linear combination of two orthogonal unit vector x and y.

$$\overrightarrow{E_0} : \begin{bmatrix} E_x \\ E_y \end{bmatrix} \begin{matrix} \leftarrow \text{x-axis} \\ \leftarrow \text{y-axis} \end{matrix}$$

where E_x and E_y can be real or complex since E is complex

$$\overrightarrow{E_0} = \hat{x}E_x + \hat{y}E_y = \hat{x}|E_x|e^{j\phi_x} + \hat{y}|E_y|e^{j\phi_y}$$

The polarization only depends on the phase difference and the magnitude ratio between the two field components.

Polarization of Light

We define

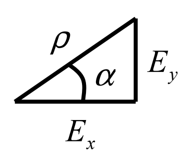
$$\varphi = \phi_y - \phi_x, \quad -\pi < \varphi \le \pi$$

and

$$\alpha = \tan^{-1}\frac{|E_y|}{|E_x|}, \quad 0 \le \alpha \le \frac{\pi}{2}$$

Because only the relative phase φ matters, we can set φ_x= 0, and take ρ to be real, then

$$\overrightarrow{E_0} = \rho \cdot \hat{e} \quad \text{with} \quad \hat{e} = \hat{x}\cos\alpha + \hat{y}e^{j\varphi}\sin\alpha$$



$$\rho = \sqrt{|E_x|^2 + |E_y|^2}$$

Polarization of Light

The space- and time-dependent **real field**

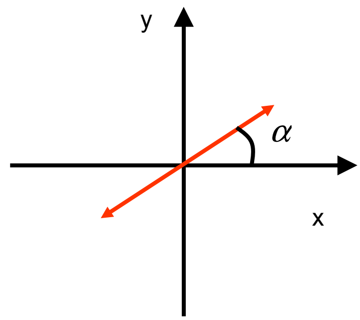
$$E(z,t) = 2\rho\left[\hat{x}\cos\alpha\cos(kz-\omega t) + \hat{y}\sin\alpha\cos(kz-\omega t+\varphi)\right]$$

Linear polarization: $\varphi = 0 \text{ or } \pi$

$$E(z,t) = 2\rho\cdot\hat{e}\cos(kz-\omega t)$$

where

$$\hat{e} = \hat{x}\cos\alpha + \hat{y}\sin\alpha$$



Polarization of Light

Circular polarization: $\varphi = \frac{\pi}{2} \text{ or } -\frac{\pi}{2}$

and $\alpha = \frac{\pi}{4} \longrightarrow |E_x| = |E_y| = \frac{\rho}{\sqrt{2}}$

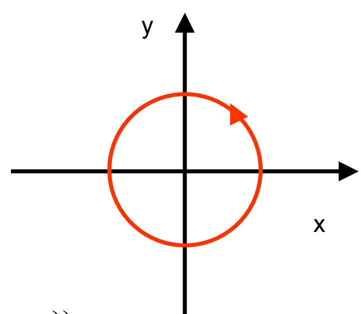
Case I: $\varphi = \frac{\pi}{2}$ (Left-circular polarization)

$$E(z,t) = \mathrm{Re}\left\{\sqrt{2}\rho\cdot\hat{e}_+\exp\{j(kz-\omega t)\}\right\}$$
$$= \sqrt{2}\rho\cdot(\hat{x}\cos(kz-\omega t) - \hat{y}\sin(kz-\omega t))$$

where

$$\hat{e}_+ = \frac{\hat{x}+j\hat{y}}{\sqrt{2}}$$

Counterclockwise

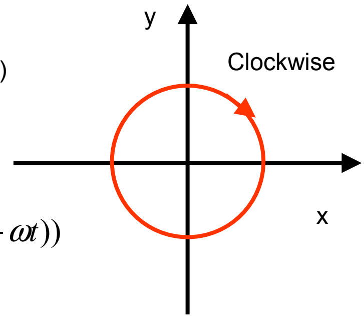


Polarization of Light

Case II: $\varphi = -\frac{\pi}{2}$ (Right-circular polarization)

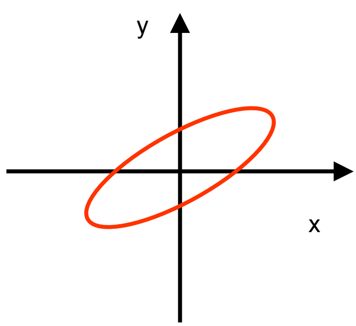
$$E(z,t) = \mathrm{Re}\left\{\sqrt{2}\rho \cdot \widehat{e}_{-} \exp\{j(kz - \omega t)\}\right\}$$
$$= \sqrt{2}\rho \cdot (\hat{x}\cos(kz - \omega t) + \hat{y}\sin(kz - \omega t))$$

where $\widehat{e}_{+} = \frac{\hat{x} - j\hat{y}}{\sqrt{2}}$

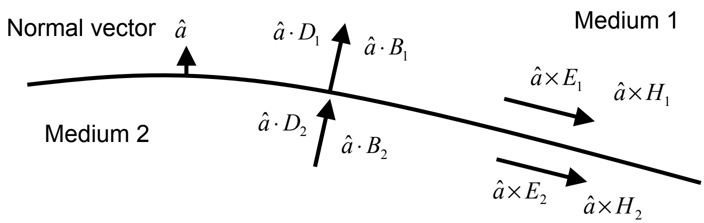


Polarization of Light

Elliptical polarization: $\varphi : otherwise$



Boundary Conditions



$$\begin{cases} \hat{a}\cdot D_1 = \hat{a}\cdot D_2 \\ \hat{a}\cdot B_1 = \hat{a}\cdot B_2 \end{cases} \quad \text{and} \quad \begin{cases} \hat{a}\times E_1 = \hat{a}\times E_2 \\ \hat{a}\times H_1 = \hat{a}\times H_2 \end{cases}$$

- **The tangential components of E and H must be continuous at the boundary**
- **The normal components of D and B must be continuous at the boundary**

Photon Nature of Light

- **The energy of a photon is determined by its frequency ν or, equivalently, its angular frequency ω**

- **The momentum of a photon is determined by its frequency λ or, equivalently, its angular frequency k**

speed : $c = \lambda\nu$

energy : $h\nu = \hbar\omega = pc$

momentum : $p = \dfrac{h\nu}{c} = \dfrac{h}{\lambda}$

Thumb of rule to calculate the photon energy for a given wavelength

$$E = h\nu = \frac{1.2398}{\lambda}\,\mu m\cdot eV$$