Average Number of Recirculations in SDL Constructions of Optical Priority Queues

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***Abstract*—In this technical report, we derive the average number of times an optical packet recirculates through the optical switch and the fiber delay lines in our previous constructions of optical priority queues (see Figure 1 in Section I) under Bernoulli arrival traffic, Bernoulli control input, and uniform priority assignment. The analytical results on the average number of recirculations are further verified through simulations. Through simulations, we also find that these analytical results are useful in choosing the number of fiber delay lines in our constructions of optical priority queues when there is a limitation on the number of times an optical packet can be recirculated through the optical switch and the fiber delay lines.**

***Index Terms*—Optical buffers, optical queues, optical switches, all-optical packet-switched networks, priority queues.**

I. Introduction

Constructing optical buffers by using optical crossbar Switches and fiber Delay Lines (SDL) for contention resolution among packets competing for the same resources in the optical domain has been well recognized as one of the feasible and promising technologies in all-optical packet-switched networks. Many SDL designs of various types of optical buffers have been proposed recently in the literature (see [1]–[3] and the references therein).

An important and practical issue that is less addressed in the SDL literature is the number of recirculations through the optical switches and the fiber delay lines. It is well known [4]–[6] that when an optical packet recirculates through the optical switches and the fiber delay lines, its signal quality is degraded as a result of many factors such as power loss when the optical packet travels through the fiber delay lines, crosstalk due to power leakage from other optical links, amplified spontaneous emission (ASE) from the Erbium doped fiber amplifiers (EDFA) that are used for boosting the signal power, and the pattern effect of the optical switches, etc. Therefore, if the number of times an optical packet recirculates through the optical switches and the fiber delay lines exceeds a certain threshold, it may not be reliably reconstructed at the destination due to severe power loss and/or serious noise accumulation even if it appears at the right place and at the right time, and such a packet is regarded as a lost packet. As such, it is important and interesting to know, on the average, how many times an optical packet recirculates through the optical switches and the fiber delay lines. Such knowledge may provide some guidelines in the SDL design of optical buffers when there is a limitation on the number of recirculations through the optical switches and the fiber delay lines.

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In this technical report, we focus on our previous SDL constructions of optical priority queues (a special type of optical buffer) in [3] by using a feedback system consisting of an $(M+1)\times(M+1)$ optical crossbar switch, a 1×2 optical crossbar switch, and M fiber delay lines with appropriately chosen delays $d_1, d_2, \ldots, d_M$ (see Figure 1). As in most works in the SDL literature, we consider the discrete-time setting in which time is slotted and synchronized, and we assume that packets are of the same size so that a packet can be transmitted within a time slot. We note that for variable-length bursts, they can be first segmented into fixed-size packets at the sources and then reassembled at the destinations.

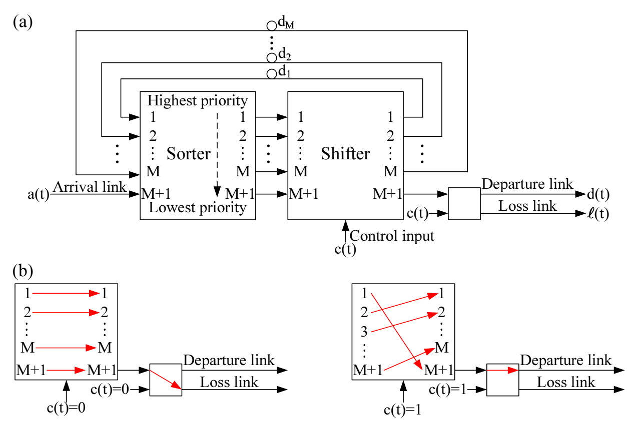

Fig. 1. (a) A construction of an optical priority queue with buffer $\sum_{i=1}^{M} d_i$ (note that the sorter and the shifter can be combined together so that they can be implemented by using a single optical crossbar switch). (b) The two possible connection patterns of the shifter and the 1×2 switch in (a).

A priority queue (see Definition 1 in [3] for a formal definition) is a network element with one arrival link, one control input, one departure link, and one loss link, and every packet in the queue has a *distinct* priority. When the control input of the priority queue is enabled, the packet with the highest priority in the queue departs from the departure link (unless the queue is empty). When the buffer of the priority queue is overflowed, the packet with the lowest priority in the queue is dropped through the loss link. Let $c(t)$ be the state of the control input of the priority queue at time t as shown in Figure 1. We say that the priority queue is enabled (resp., disabled) at time t if $c(t) = 1$ (resp., $c(t) = 0$). The main idea of our constructions of optical priority queues in [3] is to use the sorter in Figure 1(a) to sort the packets at the sorter's input links according to their priorities so that the priorities of the packets at the sorter's output links are decreasing in the indices of the sorter's output links. Then the shifter and the 1×2 optical switch in Figure 1(a) are used to route the highest priority packet to the departure link when $c(t) = 1$ (see the right-hand side of Figure 1(b)), and route the lowest priority packet to the loss link when $c(t) = 0$ (see the left-hand side of Figure 1(b)). By so doing, we showed in [3] that we achieve an exact emulation of an optical priority queue with buffer size $B = \sum_{i=1}^{M} d_i$ if we choose $d_i = d_{M+1-i} = i$ for $i = 1, 2, \ldots, m$ and $m \leq d_i = d_{M+1-i} \leq i + \sum_{j=2}^{m} \lceil ((i-M+2m-4j+1)/2)^+ \rceil$ for $i = m+1, m+2, \ldots, \lceil M/2 \rceil$, where $0 \leq m \leq \lceil M/2 \rceil$. In order to achieve the maximum buffer size that is possible under our constructions, in this technical report we choose $d_i = d_{M+1-i} = i$ for $i = 1, 2, \ldots, m$ and $d_i = d_{M+1-i} = i + \sum_{j=2}^{m} \lceil ((i-M+2m-4j+1)/2)^+ \rceil$ for $i = m+1, m+2, \ldots, \lceil M/2 \rceil$, where m is chosen as the optimal value in $\{0, 1, \ldots, \lceil M/2 \rceil\}$ that maximizes the buffer size $B = \sum_{i=1}^{M} d_i$.

This technical report is organized as follows. In Section II, we present our results on the average number of recirculations through the sorter and the shifter in Figure 1. Then we show our simulation results in Section III and conclude this technical report in Section IV.

II. Average Number of Recirculations

In this section, we derive the average number of times, N_r, an optical packet recirculates through the sorter and the shifter in our constructions of optical priority queues in Figure 1 under i.i.d. Bernoulli arrival traffic, i.i.d. Bernoulli control input, and uniform priority assignment.

To be more precise, let $a(t)$ (resp., $d(t)$, $\ell(t)$) be the number of arrival packets (resp., departure packets, lost packets) at time t, and let $q(t)$ be the number of packets stored in the buffer of the priority queue at time t (at the end of the t^{th} time slot). We derive N_r under the following assumptions: (i) The arrival process $\{a(t),\ t \geq 0\}$ is a sequence of i.i.d. Bernoulli random variables with mean α, i.e., $P(a(t) = 1) = \alpha$, and this is independent of everything else. (ii) The control input process $\{c(t),\ t \geq 0\}$ is a sequence of i.i.d. Bernoulli random variables with mean β, i.e., $P(c(t) = 1) = \beta$, and this is independent of everything else. (iii) The priority of an arrival packet is uniformly distributed with respect to those of the packets in the priority queue when it arrives, and this is also independent of everything else. Specifically, if there is an arrival packet at time t, then its priority is uniformly distributed over $\{1, 2, \ldots, q(t-1), q(t-1)+1\}$, and the priorities of the $q(t-1)$ packets stored in the buffer at time $t-1$ are updated accordingly so that their relative priority order is not changed. In other words, if the priority of the arrival packet at time t is j, where $1 \leq j \leq q(t-1)+1$, then the priority j' packet stored in the buffer at time $t-1$ still has the same priority j' for $j' = 1, 2, \ldots, j-1$, but has an updated priority $j'+1$ for $j' = j, j+1, \ldots, q(t-1)$. Such an operation is also known as a Push-In–First-Out (PIFO) queue in the literature.

We remark that the i.i.d. Bernoulli arrival traffic in (i) is a commonly adopted assumption in the literature, and the uniform priority assignment in (iii) is a reasonable assumption when no further information about the arrival traffic, except its arrival rate, is available. Regarding the i.i.d. Bernoulli control input in (ii), we note that the control input is for enabling/disabling the priority queue for the usage of the departure link (the departure link could be viewed as resources that are also shared by some other network elements), and is regulated by certain resource management or congestion control schemes. In the case that only the arrival rate is available, it is less complex and less costly for the resource manager to simply provide an enabling rate β that meets certain requirement of quality of service.

In the rest of this technical report, we denote $\overline{\alpha} = 1 - \alpha$ and $\overline{\beta} = 1 - \beta$ for ease of presentation. Under the assumptions in (i) and (ii), we can see that the queue length process $\{q(t),\ t \geq 0\}$ is a discrete-time birth-death process with the following state transition probabilities:

$$\begin{aligned} & P(q(t) = i+1 | q(t-1) = i) \\ & \quad = P(a(t) = 1, c(t) = 0 | q(t-1) = i) \\ & \quad = P(a(t) = 1, c(t) = 0) = \alpha\overline{\beta}, \text{ for } i = 0, 1, \ldots, B-1 \quad (1) \\ & P(q(t) = i-1 | q(t-1) = i) \\ & \quad = P(a(t) = 0, c(t) = 1 | q(t-1) = i) \\ & \quad = P(a(t) = 0, c(t) = 1) = \overline{\alpha}\beta, \text{ for } i = 1, 2, \ldots, B, \quad (2) \\ & P(q(t) = i | q(t-1) = i) \\ & \quad = P(a(t) = 0, c(t) = 0) + P(a(t) = 1, c(t) = 1) \\ & \quad = \overline{\alpha} \cdot \overline{\beta} + \alpha\beta, \text{ for } i = 1, 2, \ldots, B-1, \quad (3) \\ & P(q(t) = 0 | q(t-1) = 0) \\ & \quad = P(c(t) = 1) + P(a(t) = 0, c(t) = 0) = \beta + \overline{\alpha} \cdot \overline{\beta}, \quad (4) \\ & P(q(t) = B | q(t-1) = B) \\ & \quad = P(c(t) = 0) + P(a(t) = 1, c(t) = 1) = \overline{\beta} + \alpha\beta, \quad (5) \\ & P(q(t) = j | q(t-1) = i) = 0, \text{ for the other } i \text{ and } j. \quad (6) \end{aligned}$$

Let P be the transition probability matrix specified by (1)–(6), then the unique steady state probabilities $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_B)$ for the birth-death process $\{q(t),\ t \geq 0\}$ can be obtained by solving $\boldsymbol{\pi} = \boldsymbol{\pi} P$ and the result is $\pi_i = \rho^i \pi_0$ for $0 \leq i \leq B$, where $\rho = \frac{\alpha\overline{\beta}}{\beta\overline{\alpha}}$ and

$$\pi_0 = \begin{cases} \frac{1}{B+1}, & \text{if } \alpha = \beta, \\ \frac{1-\rho}{1-\rho^{B+1}}, & \text{if } \alpha \neq \beta. \end{cases} \quad (7)$$

In the following theorem, we derive a closed-form expression for N_r for the case that $\alpha = \beta$ under the assumptions in (i) and (ii), and give an approximation expression for N_r for the case that $\alpha \neq \beta$ under the assumptions in (i)–(iii) (we note that N_r still can be computed for the case that $\alpha \neq \beta$ under the assumptions in (i) and (ii), even though we are not able to obtain a closed-form expression for N_r in this case).

Theorem 1 *(a) If $\alpha = \beta$, then $N_r = \frac{M}{2\alpha} + 1$ under the assumptions in (i) and (ii).*

(b) If $\alpha \neq \beta$, then $N_r \approx (\pi_0(\beta \sum_{i=0}^{B-1} \frac{\rho^i}{i+1} + \frac{\rho^B}{B+1}))^{-1}$ under the assumptions in (i)–(iii), where $\rho = \frac{\alpha\overline{\beta}}{\beta\overline{\alpha}}$ and $\pi_0 = \frac{1-\rho}{1-\rho^{B+1}}$.

Proof. (a) From (7), we see that the average number of packets L_q stored in the buffer in steady state is given by $L_q = \sum_{i=0}^{B} i \cdot \pi_i = \sum_{i=0}^{B} i \cdot \frac{1}{B+1} = \frac{B}{2}$. As the arrival rate λ in steady state is given by $\lambda = \lim_{t\to\infty} E[a(t)] = \alpha$, it follows from Little's formula [7] that the average waiting time W_q of a packet in the queue in steady state is given by $W_q = \frac{L_q}{\lambda} = \frac{B}{2\alpha}$.

In Appendix A, we will show that the average recirculation time (per recirculation) T_r of an optical packet through the fiber delay lines in steady state is given by $T_r = \frac{B/2}{M/2} = \frac{B}{M}$, where $M/2$ is the average number of packets routed into the M fiber delay lines at a given time slot in steady state and $B/2$ is the average recirculation time (per recirculation) through the fiber delay lines of the packets routed into the M fiber delay lines at a given time slot in steady state. As the number of times an optical packet recirculates through the sorter and the shifter is always one more than that through the fiber delay lines in our constructions, it then follows that the average number of recirculations through the sorter and the shifter is given by $N_r = \frac{W_q}{T_r} + 1 = \frac{B/(2\alpha)}{B/M} + 1 = \frac{M}{2\alpha} + 1$.

We remark that for the case that $\alpha \neq \beta$, T_r can be computed as described at the end of Appendix A, and we can still compute N_r as $N_r = \frac{W_q}{T_r} + 1$ (even though we are not able to obtain a closed-form expression for N_r in this case).

(b) Suppose that there is an arrival packet at time t in steady state. Call this packet the tagged packet and let $\gamma(t)$ be the priority of the tagged packet ($\gamma(t) = i$ means that the tagged packet is the i^{th} highest priority packet). If $q(t-1) = i$, where $0 \le i \le B-1$, then the tagged packet is routed to the departure link with probability $P(\gamma(t) = 1, c(t) = 1) = \beta/(i+1)$, and is routed to one of the fiber delay lines with probability $1 - \beta/(i+1)$. On the other hand, if $q(t-1) = B$, then the tagged packet is routed to the departure link with probability $P(\gamma(t) = 1, c(t) = 1) = \beta/(B+1)$, is routed to the loss link with probability $P(\gamma(t) = B+1, c(t) = 0) = \overline{\beta}/(B+1)$, and is routed to one of the fiber delay lines with probability $1 - \beta/(B+1) - \overline{\beta}/(B+1) = B/(B+1)$.

Let $X_r(t)$ be the number of times that the tagged packet recirculates through the sorter and the shifter, and let $X_r^{(i)}(t)$ be the number of times that the tagged packet recirculates through the sorter and the shifter conditioned on $q(t-1) = i$ and the tagged packet being routed to one of the fiber delay lines for $0 \le i \le B$. Clearly, we have

$$N_r = E[X_r(t)] = \sum_{i=0}^{B} P(q(t-1) = i) E[X_r(t) | q(t-1) = i]$$

$$= \sum_{i=0}^{B-1} \pi_i \left(\frac{\beta}{i+1} \cdot 1 + \left(1 - \frac{\beta}{i+1}\right) \cdot E[X_r^{(i)}(t)] \right)$$

$$+ \pi_B \left(\frac{\beta}{B+1} \cdot 1 + \frac{\overline{\beta}}{B+1} \cdot 1 + \frac{B}{B+1} \cdot E[X_r^{(B)}(t)] \right).$$

In the case that the tagged packet is routed to one of the fiber delay lines, we make the approximation that it behaves like a new arrival packet when it comes out of that fiber delay line and reappears at the inputs of the sorter. As such, we can approximate $E[X_r^{(i)}(t)] \approx N_r + 1$ for all i, and it follows that $N_r \approx N_r + 1 - N_r \beta \sum_{i=0}^{B-1} \frac{\pi_i}{i+1} - N_r \frac{\pi_B}{B+1}$. This leads to $N_r \approx (\beta \sum_{i=0}^{B-1} \frac{\pi_i}{i+1} + \frac{\pi_B}{B+1})^{-1} = (\pi_0(\beta \sum_{i=0}^{B-1} \frac{\rho^i}{i+1} + \frac{\rho^B}{B+1}))^{-1}$. ■

III. Simulation Results

In Figure 2 and Figure 3, we show our simulation results. In our simulations, the simulation time is 10^8 time slots. Note that although the results in this technical report hold for arbitrary α and β, in practice it is more reasonable to choose $\alpha \le \beta$ (the arrival rate is less than or equal to the service rate).

For the case that $\alpha = \beta = 0.9$, we can see from Figure 2(a) that the analytical result on N_r in Theorem 1(a) matches very well with the simulation results. In Figure 2(b), Y_r is the number of recirculations of a packet through the sorter and the shifter in Figure 1. We can see from Figure 2(b) that $P(Y_r > C_i(\frac{M}{2\alpha} + 1)) < 10^{-i}$ for $i = 2, 3, 4$, where $C_2 = 5$, $C_3 = 10$, and $C_4 = 18$. When an optical packet recirculating through the sorter and the shifter more than R times is regarded as a lost packet and we can tolerate a packet loss probability of 10^{-i}, this tells us that we need to choose M such that $M \le \lfloor 2\alpha(\frac{R}{C_i} - 1) \rfloor$ in Figure 1 for $i = 2, 3, 4$.

For the case that $\alpha = 0.9$ and $\beta = 0.95$, we see from Figure 3(a) that our approximation result on N_r in Theorem 1(b) is quite good as the approximation values are very close to the simulation results. As in this case we have $\alpha < \beta$, the queue size is small with high probability and it follows that most of the time only a few fiber delay lines are used for recirculating packets. As such, N_r and $P(Y_r > x)$ will be approximately the same for sufficiently large values of M as can be seen from the results for different values of M in Figure 3.

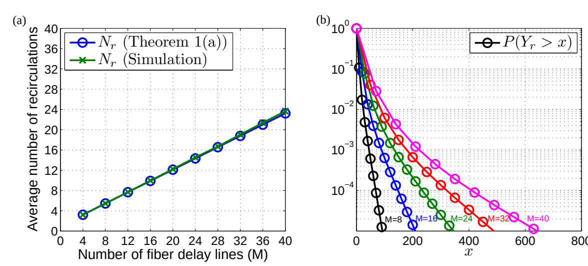

Fig. 2. $\alpha = \beta = 0.9$: (a) Average number of recirculations. (b) Complementary distribution of the number of recirculations.

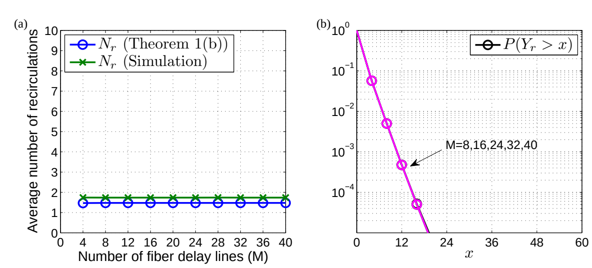

Fig. 3. $\alpha = 0.9$ and $\beta = 0.95$: (a) Average number of recirculations. (b) Complementary distribution of the number of recirculations.

IV. Conclusion

In this technical report, we derived the average number of recirculations through the optical switch and the fiber delay lines in our previous SDL constructions of optical priority queues in Figure 1 under i.i.d. Bernoulli arrival traffic, i.i.d. Bernoulli control input, and uniform priority assignment. The results are useful in choosing the number of fiber delay lines when there is a limitation on the number of recirculations through the optical switch and the fiber delay lines in Figure 1.

Appendix A
Proof of $T_r = B/M$ in the Proof of Theorem 1

To prove the claim that $T_r = B/M$ in the proof of Theorem 1, we view a fiber delay line with delay d as a *sequential* buffer that consists of d cells with each cell capable of holding one packet. In Figure 4, we show the cells of the fiber delay lines. Note that we index the cells from the *input* of a fiber delay line in Figure 4, namely, the $(i,j)^{\text{th}}$ cell is the j^{th} cell from the input of the i^{th} fiber delay line for $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, d_i$.

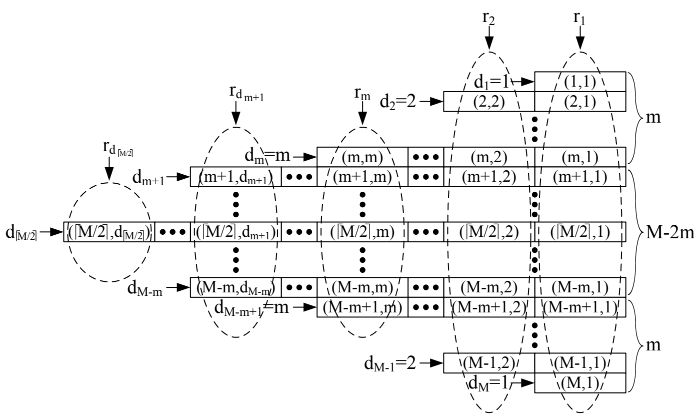

Fig. 4. The cells of the fiber delay lines.

Let $\mathbf{s}(t) = (s_1(t), s_2(t), \ldots, s_{d_{\lceil M/2 \rceil}}(t))$, where $s_j(t)$ is the number of packets stored in the cells of the j^{th} column in Figure 4 at time t (at the end of the t^{th} time slot) for $1 \le j \le d_{\lceil M/2 \rceil}$. From the operations of our constructions of optical priority queues as described in Section I, we can see that $\mathbf{s}(t)$ only depends on $\mathbf{s}(t-1)$, $a(t)$, and $c(t)$. As such, the process $\{\mathbf{s}(t),\ t \ge 0\}$ is a discrete-time Markov chain. For $\mathbf{s} = (s_1, s_2, \ldots, s_{d_{\lceil M/2 \rceil}})$ and $\mathbf{s}' = (s'_1, s'_2, \ldots, s'_{d_{\lceil M/2 \rceil}})$, let $\tilde{P}_{\mathbf{s},\mathbf{s}'} = P(\mathbf{s}(t) = \mathbf{s}' | \mathbf{s}(t-1) = \mathbf{s})$ be the state transition probability from state $\mathbf{s}$ at time $t-1$ to state $\mathbf{s}'$ at time t, and let $\tilde{\pi}_{\mathbf{s}} = \lim_{t \to \infty} P(\mathbf{s}(t) = \mathbf{s})$ be the steady state probability of the Markov chain $\{\mathbf{s}(t),\ t \ge 0\}$ being in state $\mathbf{s}$.

Let $\mathbf{r} = (r_1, r_2, \ldots, r_{d_{\lceil M/2 \rceil}})$, where r_j is the number of cells of the j^{th} column in Figure 4 for $1 \le j \le d_{\lceil M/2 \rceil}$. Clearly, we have $r_1 = M$. Note that as we have $1 = d_1 < d_2 < \cdots < d_{\lceil M/2 \rceil}$, there exists a unique integer $1 \le i_j \le \lceil M/2 \rceil$ such that $d_{i_j - 1} < j \le d_{i_j}$ for $1 \le j \le d_{\lceil M/2 \rceil}$. Since $d_i = d_{M+1-i}$ for $1 \le i \le \lceil M/2 \rceil$, we immediately see that

$$r_j = (M + 1 - i_j) - i_j + 1 = M + 2 - 2i_j, \quad \text{for } 1 \le j \le d_{\lceil M/2 \rceil}. \tag{8}$$

For a state $\mathbf{s} = (s_1, s_2, \ldots, s_{d_{\lceil M/2 \rceil}})$, we denote $\overline{\mathbf{s}} = \mathbf{r} - \mathbf{s}$, namely, $(\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_{d_{\lceil M/2 \rceil}}) = (r_1 - s_1, r_2 - s_2, \ldots, r_{d_{\lceil M/2 \rceil}} - s_{d_{\lceil M/2 \rceil}})$.

We need the following two lemmas for our proof.

Lemma 2 *$\tilde{P}_{\mathbf{s},\mathbf{s}'} = \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}'}}$ for all states $\mathbf{s}$ and $\mathbf{s}'$.*

Proof. Suppose that $\mathbf{s} = (s_1, s_2, \ldots, s_{d_{\lceil M/2 \rceil}})$. Given that $\mathbf{s}(t-1) = \mathbf{s}$ (resp., $\mathbf{s}(t-1) = \overline{\mathbf{s}}$), let $n(\mathbf{s})$ (resp., $n(\overline{\mathbf{s}})$) be the number of packets stored in the M cells $(1, d_1), (2, d_2), \ldots, (M, d_M)$ (the last cells of the M fiber delay lines) in Figure 4 at time $t-1$ (at the end of the $(t-1)^{\text{th}}$ time slot). Note that these $n(\mathbf{s})$ (resp., $n(\overline{\mathbf{s}})$) packets will appear at the input links of the sorter in Figure 1 at the beginning of the t^{th} time slot. Given that $\mathbf{s}(t-1) = \mathbf{s}$ (resp., $\mathbf{s}(t-1) = \overline{\mathbf{s}}$) and $(a(t), c(t)) = (a, c)$, let $\mathbf{s}^{a,c} = ((s^{a,c})_1, (s^{a,c})_2, \ldots, (s^{a,c})_{d_{\lceil M/2 \rceil}})$ (resp., $\overline{\mathbf{s}}^{a,c} = ((\overline{s}^{a,c})_1, (\overline{s}^{a,c})_2, \ldots, (\overline{s}^{a,c})_{d_{\lceil M/2 \rceil}})$) be the state at time t for $(a, c) = (0,0), (0,1), (1,0), (1,1)$. Since the packets stored in the cells of the $(j-1)^{\text{th}}$ column at time $t-1$ simply move forward to the cells of the j^{th} column during the t^{th} time slot for $2 \le j \le d_{\lceil M/2 \rceil}$ (except the packets in the last cells of the M fiber delay lines), we can see that

$$(s^{0,0})_j = (s^{0,1})_j = (s^{1,0})_j = (s^{1,1})_j, \ 2 \le j \le d_{\lceil M/2 \rceil}. \tag{9}$$

Given that $\mathbf{s}(t-1) = \mathbf{s}$, let $p_{i,j}(\mathbf{s}) = 1$ if there is a packet stored in cell (i,j) in Figure 4 at time $t-1$ (at the end of the $(t-1)^{\text{th}}$ time slot) and $p_{i,j}(\mathbf{s}) = 0$ otherwise for $1 \le i \le M$ and $1 \le j \le d_i$. Since the packets from the shifter's output links are routed to the fiber delay lines "consecutively" starting from the first fiber delay line in our constructions of optical priority queues, we can see that, given that $\mathbf{s}(t-1) = \mathbf{s}$, the s_j packets stored in the cells of the j^{th} column in Figure 4 at time $t-1$ are in the first s_j cells of the j^{th} column for $1 \le j \le d_{\lceil M/2 \rceil}$. It follows that

$$p_{i,j}(\mathbf{s}) = \begin{cases} 1, & \text{if } i_j \le i \le i_j + s_j - 1 \text{ and } 1 \le j \le d_{\lceil M/2 \rceil}, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

We claim that

$$p_{i,j}(\mathbf{s}) + p_{M+1-i,j}(\overline{\mathbf{s}}) = 1, \quad \text{for } i_j \le i \le M + 1 - i_j \text{ and } 1 \le j \le d_{\lceil M/2 \rceil}. \tag{11}$$

To see this, if $p_{i,j}(\mathbf{s}) = 0$, then we have from (10) that $i \ge i_j + s_j$. It follows from (8) that $M + 1 - i \le M + 1 - i_j - s_j = i_j + r_j - 1 - s_j = i_j + \overline{s}_j - 1$, and hence we see from (10) that $p_{M+1-i,j}(\overline{\mathbf{s}}) = 1$. On the other hand, if $p_{i,j}(\mathbf{s}) = 1$, then we have $i \le i_j + s_j - 1$ and it follows that $M + 1 - i \ge M + 2 - i_j - s_j = i_j + r_j - s_j = i_j + \overline{s}_j$, implying that $p_{M+1-i,j}(\overline{\mathbf{s}}) = 0$.

Similarly, given that $\mathbf{s}(t-1) = \mathbf{s}$ and $(a(t), c(t)) = (a, c)$, the state at time t is $\mathbf{s}^{a,c}$ and we let $p'_{i,j}(\mathbf{s}^{a,c}) = 1$ if there is a packet stored in cell (i,j) in Figure 4 at time t (at the end of the t^{th} time slot) and $p'_{i,j}(\mathbf{s}^{a,c}) = 0$ otherwise for $1 \le i \le M$ and $1 \le j \le d_i$, where $(a, c) = (0,0), (0,1), (1,0), (1,1)$. Again, since the packets stored in the cells of the $(j-1)^{\text{th}}$ column at time $t-1$ simply move forward to the cells of the j^{th} column during the t^{th} time slot for $2 \le j \le d_{\lceil M/2 \rceil}$ (except the packets in the last cells of the M fiber delay lines), it is easy to deduce from (11) that

$$p'_{i,j}(\mathbf{s}^{a,c}) + p'_{M+1-i,j}(\overline{\mathbf{s}}^{a',c'}) = 1, \quad \text{for } i_j \le i \le M+1-i_j \text{ and } 2 \le j \le d_{\lceil M/2 \rceil}, \quad \text{and for all } a,c,a',c'. \tag{12}$$

It then follows from the definition of $\mathbf{s}^{a,c}$ and $\overline{\mathbf{s}}^{a',c'}$, the definition of $p'_{i,j}(\mathbf{s}^{a,c})$ and $p'_{i,j}(\overline{\mathbf{s}}^{a',c'})$, (12), and (8) that

$$\begin{aligned}
&(s^{a,c})_j + (\overline{s}^{a',c'})_j \\
&= \sum_{i=i_j}^{M+1-i_j} p'_{i,j}(\mathbf{s}^{a,c}) + \sum_{i=i_j}^{M+1-i_j} p'_{i,j}(\overline{\mathbf{s}}^{a',c'}), \\
&= \sum_{i=i_j}^{M+1-i_j} p'_{i,j}(\mathbf{s}^{a,c}) + \sum_{i=i_j}^{M+1-i_j} p'_{M+1-i,j}(\overline{\mathbf{s}}^{a',c'}), \\
&= \sum_{i=i_j}^{M+1-i_j} 1 = M+2-2i_j = r_j, \\
&\qquad \text{for } 2 \le j \le d_{\lceil M/2 \rceil} \text{ and all } a,c,a',c'.
\end{aligned} \tag{13}$$

Note that if $j = d_i$, then we have $i_j = i$ for the case that $1 \le i \le \lceil M/2 \rceil$, and we have $i_j = M+1-i$ (as $d_i = d_{M+1-i}$) for the case that $\lceil M/2 \rceil + 1 \le i \le M$. Thus, we have from (11) that

$$p_{i,d_i}(\mathbf{s}) + p_{M+1-i,d_i}(\overline{\mathbf{s}}) = 1, \text{ for } 1 \le i \le M. \tag{14}$$

As such, we see from the definition of $n(\mathbf{s})$ and $n(\overline{\mathbf{s}})$, the definition of $p_{i,j}(\mathbf{s})$ and $p_{i,j}(\overline{\mathbf{s}})$, $d_i = d_{M+1-i}$, and (14) that

$$\begin{aligned}
n(\mathbf{s}) + n(\overline{\mathbf{s}}) &= \sum_{i=1}^{M} p_{i,d_i}(\mathbf{s}) + \sum_{i=1}^{M} p_{i,d_i}(\overline{\mathbf{s}}), \\
&= \sum_{i=1}^{M} p_{i,d_i}(\mathbf{s}) + \sum_{i=1}^{M} p_{i,d_{M+1-i}}(\overline{\mathbf{s}}), \\
&= \sum_{i=1}^{M} p_{i,d_i}(\mathbf{s}) + \sum_{i=1}^{M} p_{M+1-i,d_i}(\overline{\mathbf{s}}), \\
&= \sum_{i=1}^{M} 1 = M.
\end{aligned} \tag{15}$$

As it is clear that $0 \le n(\mathbf{s}) \le M$, in the following we discuss the three cases $1 \le n(\mathbf{s}) \le M-1$, $n(\mathbf{s}) = 0$, and $n(\mathbf{s}) = M$ separately.

Case 1. $1 \le n(\mathbf{s}) \le M-1$: In this case, we first show that

$$(s^{0,0})_1 = (s^{1,1})_1 = n(\mathbf{s}), \tag{16}$$
$$(s^{0,1})_1 = n(\mathbf{s}) - 1, \tag{17}$$
$$(s^{1,0})_1 = n(\mathbf{s}) + 1. \tag{18}$$

To see this, suppose that $\mathbf{s}(t-1) = \mathbf{s}$. If $(a(t), c(t)) = (0,0)$, then the $n(\mathbf{s})$ packets from the last cells of the fiber delay lines that appear at the input links of the sorter at the beginning of the t^{th} time slot will be stored in the first $n(\mathbf{s})$ cells of first column in Figure 4 at time t, and we have $(s^{0,0})_1 = n(\mathbf{s})$. If $(a(t), c(t)) = (1,1)$, then there is one arrival packet and one departure packet at time t, and we still have $n(\mathbf{s})$ packets stored in the first $n(\mathbf{s})$ cells of first column in Figure 4 at time t, implying that $(s^{1,1})_1 = n(\mathbf{s})$. If $(a(t), c(t)) = (0,1)$, then there is one departure packet at time t (note that we have $n(\mathbf{s}) \ge 1$ in this case) and there are $n(\mathbf{s}) - 1$ packets stored in the first $n(\mathbf{s}) - 1$ cells of first column in Figure 4 at time t, and thus we have $(s^{0,1})_1 = n(\mathbf{s}) - 1$. Finally, if $(a(t), c(t)) = (1,0)$, then there is one arrival packet at time t and there are $n(\mathbf{s}) + 1$ packets stored in the first $n(\mathbf{s}) + 1$ cells of first column in Figure 4 at time t (note that we have $n(\mathbf{s}) \le M-1$ in this case), and hence we have $(s^{1,0})_1 = n(\mathbf{s}) + 1$.

It follows from (9) and (16)–(18) that $\mathbf{s}^{0,0} = \mathbf{s}^{1,1}$ and the three states $\mathbf{s}^{0,0}$, $\mathbf{s}^{0,1}$, and $\mathbf{s}^{1,0}$ are distinct. Therefore, we have

$$\begin{aligned}
\tilde{P}_{\mathbf{s},\mathbf{s}^{0,0}} &= P((a(t),c(t)) = (0,0)) + P((a(t),c(t)) = (1,1)) \\
&= \overline{\alpha} \cdot \overline{\beta} + \alpha\beta,
\end{aligned} \tag{19}$$
$$\tilde{P}_{\mathbf{s},\mathbf{s}^{0,1}} = P((a(t),c(t)) = (0,1)) = \overline{\alpha}\beta, \tag{20}$$
$$\tilde{P}_{\mathbf{s},\mathbf{s}^{1,0}} = P((a(t),c(t)) = (1,0)) = \alpha\overline{\beta}, \tag{21}$$
$$\tilde{P}_{\mathbf{s},\mathbf{s}'} = 0, \text{ if } \mathbf{s}' \notin \{\mathbf{s}^{0,0}, \mathbf{s}^{0,1}, \mathbf{s}^{1,0}\}. \tag{22}$$

Since in this case we have $1 \le n(\mathbf{s}) \le M-1$, we see from (15) that $1 \le n(\overline{\mathbf{s}}) \le M-1$ and hence (16)–(22) also hold for the state $\overline{\mathbf{s}}$.

Now we show that

$$\overline{\mathbf{s}^{a,c}} = \overline{\mathbf{s}}^{1-a,1-c}, \text{ for } (a,c) = (0,0),(0,1),(1,0),(1,1). \tag{23}$$

In other words, we will show that $(s^{a,c})_j + (\overline{s}^{1-a,1-c})_j = r_j$ for $1 \le j \le d_{\lceil M/2 \rceil}$ and $(a,c) = (0,0),(0,1),(1,0),(1,1)$. From (13), it suffices to prove that $(s^{a,c})_1 + (\overline{s}^{1-a,1-c})_1 = r_1$ for $(a,c) = (0,0),(0,1),(1,0),(1,1)$. For this, we see from (16)–(18), (15), and $M = r_1$ that

$$\begin{aligned}
&(s^{0,0})_1 + (\overline{s}^{1,1})_1 = n(\mathbf{s}) + n(\overline{\mathbf{s}}) = M = r_1, \\
&(s^{0,1})_1 + (\overline{s}^{1,0})_1 = (n(\mathbf{s}) - 1) + (n(\overline{\mathbf{s}}) + 1) = M = r_1, \\
&(s^{1,0})_1 + (\overline{s}^{0,1})_1 = (n(\mathbf{s}) + 1) + (n(\overline{\mathbf{s}}) - 1) = M = r_1, \\
&(s^{1,1})_1 + (\overline{s}^{0,0})_1 = n(\mathbf{s}) + n(\overline{\mathbf{s}}) = M = r_1.
\end{aligned}$$

Therefore, we have from $\mathbf{s}^{0,0} = \mathbf{s}^{1,1}$, (23), (19)–(22), and $\alpha = \beta$ that

$$\begin{aligned}
\tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}^{0,0}}} &= \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}^{1,1}}} = \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}}^{0,0}} = \overline{\alpha} \cdot \overline{\beta} + \alpha\beta \\
&= \tilde{P}_{\mathbf{s},\mathbf{s}^{0,0}} = \tilde{P}_{\mathbf{s},\mathbf{s}^{1,1}},
\end{aligned} \tag{24}$$
$$\tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}^{0,1}}} = \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}}^{1,0}} = \alpha\overline{\beta} = \overline{\alpha}\beta = \tilde{P}_{\mathbf{s},\mathbf{s}^{0,1}}, \tag{25}$$
$$\tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}^{1,0}}} = \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}}^{0,1}} = \overline{\alpha}\beta = \alpha\overline{\beta} = \tilde{P}_{\mathbf{s},\mathbf{s}^{1,0}}, \tag{26}$$
$$\tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}'}} = 0 = \tilde{P}_{\mathbf{s},\mathbf{s}'}, \text{ if } \mathbf{s}' \notin \{\mathbf{s}^{0,0}, \mathbf{s}^{0,1}, \mathbf{s}^{1,0}\}. \tag{27}$$

By combining (24)–(27), we have $\tilde{P}_{\mathbf{s},\mathbf{s}'} = \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}'}}$ for all $\mathbf{s}'$ in this case.

Case 2. $n(\mathbf{s}) = 0$: In this case, we first show that

$$(s^{0,0})_1 = (s^{1,1})_1 = (s^{0,1})_1 = 0, \tag{28}$$
$$(s^{1,0})_1 = 1. \tag{29}$$

To see this, suppose that $\mathbf{s}(t-1) = \mathbf{s}$. If $(a(t), c(t)) = (0,0)$ or $(a(t), c(t)) = (1,1)$, then we have $(s^{0,0})_1 = (s^{1,1})_1 = n(\mathbf{s}) = 0$ by following the same argument as in Case 1. If $(a(t), c(t)) = (0,1)$, then there are no packets stored in the cells of first column in Figure 4 at time t since there are no packets from the last cells of the fiber delay lines appearing at the input links of the sorter at the beginning of the t^{th} time slot

$(n(\mathbf{s}) = 0)$ and there are no arrival packets at time t $(a(t) = 0)$ even though the control input is enabled $(c(t) = 1)$, and thus we have $(s^{0,1})_1 = 0$. Finally, if $(a(t), c(t)) = (1, 0)$, then we have $(s^{1,0})_1 = n(\mathbf{s}) + 1 = 1$ by following the same argument as in Case 1.

It follows from (9), (28), and (29) that $\mathbf{s}^{0,0} = \mathbf{s}^{1,1} = \mathbf{s}^{0,1}$ and the two states $\mathbf{s}^{0,0}$ and $\mathbf{s}^{1,0}$ are distinct. Therefore, we have

$$\begin{aligned}\tilde{P}_{\mathbf{s},\mathbf{s}^{0,0}} &= P((a(t), c(t)) = (0, 0)) + P((a(t), c(t)) = (1, 1)) \\ &\quad + P((a(t), c(t)) = (0, 1)) \\ &= \overline{\alpha} \cdot \overline{\beta} + \alpha\beta + \overline{\alpha}\beta, \qquad (30)\\ \tilde{P}_{\mathbf{s},\mathbf{s}^{1,0}} &= P((a(t), c(t)) = (1, 0)) = \alpha\overline{\beta}, \qquad (31)\\ \tilde{P}_{\mathbf{s},\mathbf{s}'} &= 0, \text{ if } \mathbf{s}' \notin \{\mathbf{s}^{0,0}, \mathbf{s}^{1,0}\}. \qquad (32)\end{aligned}$$

Next, we show that

$$\begin{aligned}(\overline{s}^{0,0})_1 = (\overline{s}^{1,1})_1 = (\overline{s}^{1,0})_1 &= M, \qquad (33)\\ (\overline{s}^{0,1})_1 &= M - 1. \qquad (34)\end{aligned}$$

To see this, suppose that $\mathbf{s}(t-1) = \overline{\mathbf{s}}$. Since in this case we have $n(\mathbf{s}) = 0$, we see from (15) that $n(\overline{\mathbf{s}}) = M$. If $(a(t), c(t)) = (0, 0)$ or $(a(t), c(t)) = (1, 1)$, then we have $(\overline{s}^{0,0})_1 = (\overline{s}^{1,1})_1 = n(\overline{\mathbf{s}}) = M$ by following the same argument as in Case 1. If $(a(t), c(t)) = (1, 0)$, then there is a packet dropped from the loss link at time t since there are M packets from the last cells of the fiber delay lines appearing at the input links of the sorter at the beginning of the t^{th} time slot $(n(\overline{\mathbf{s}}) = M)$, there is one arrival packet at time t $(a(t) = 1)$, and the control input is disabled $(c(t) = 0)$. It follows that there are M packets stored in the cells of first column in Figure 4 at time t, and we have $(\overline{s}^{1,0})_1 = M$. Finally, if $(a(t), c(t)) = (0, 1)$, then we have $(\overline{s}^{0,1})_1 = n(\overline{\mathbf{s}}) - 1 = M - 1$ by following the same argument as in Case 1.

It follows from (9), (33), and (34) that $\overline{\mathbf{s}}^{0,0} = \overline{\mathbf{s}}^{1,1} = \overline{\mathbf{s}}^{1,0}$ and the two states $\overline{\mathbf{s}}^{0,0}$ and $\overline{\mathbf{s}}^{0,1}$ are distinct. Therefore, we have

$$\begin{aligned}\tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}}^{0,0}} &= P((a(t), c(t)) = (0, 0)) + P((a(t), c(t)) = (1, 1)) \\ &\quad + P((a(t), c(t)) = (1, 0)) \\ &= \overline{\alpha} \cdot \overline{\beta} + \alpha\beta + \alpha\overline{\beta}, \qquad (35)\\ \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}}^{0,1}} &= P((a(t), c(t)) = (0, 1)) = \overline{\alpha}\beta, \qquad (36)\\ \tilde{P}_{\overline{\mathbf{s}},\mathbf{s}'} &= 0, \text{ if } \mathbf{s}' \notin \{\overline{\mathbf{s}}^{0,0}, \overline{\mathbf{s}}^{0,1}\}. \qquad (37)\end{aligned}$$

Now we show that (23) also holds in this case. As in Case 1, it suffices to prove that $(s^{a,c})_1 + (\overline{s}^{1-a,1-c})_1 = r_1$ for $(a, c) = (0, 0), (0, 1), (1, 0), (1, 1)$. For this, we see from (28), (29), (33), (34), and $M = r_1$ that

$$\begin{aligned}(s^{0,0})_1 + (\overline{s}^{1,1})_1 &= 0 + M = r_1,\\ (s^{0,1})_1 + (\overline{s}^{1,0})_1 &= 0 + M = r_1,\\ (s^{1,0})_1 + (\overline{s}^{0,1})_1 &= 1 + (M - 1) = r_1,\\ (s^{1,1})_1 + (\overline{s}^{0,0})_1 &= 0 + M = r_1.\end{aligned}$$

Therefore, we have from $\mathbf{s}^{0,0} = \mathbf{s}^{1,1} = \mathbf{s}^{0,1}$, (23), (30)–(32), (35)–(37), and $\alpha = \beta$ that

$$\begin{aligned}\tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}^{0,0}}} &= \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}^{0,1}}} = \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}^{1,1}}} = \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}}^{0,0}} = \overline{\alpha} \cdot \overline{\beta} + \alpha\beta + \alpha\overline{\beta} \\ &= \overline{\alpha} \cdot \overline{\beta} + \alpha\beta + \overline{\alpha}\beta = \tilde{P}_{\mathbf{s},\mathbf{s}^{0,0}} = \tilde{P}_{\mathbf{s},\mathbf{s}^{0,1}} = \tilde{P}_{\mathbf{s},\mathbf{s}^{1,1}}, \quad (38)\\ \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}^{1,0}}} &= \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}}^{0,1}} = \overline{\alpha}\beta = \alpha\overline{\beta} = \tilde{P}_{\mathbf{s},\mathbf{s}^{1,0}}, \qquad (39)\\ \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}'}} &= 0 = \tilde{P}_{\mathbf{s},\mathbf{s}'}, \text{ if } \mathbf{s}' \notin \{\mathbf{s}^{0,0}, \mathbf{s}^{1,0}\}. \qquad (40)\end{aligned}$$

By combining (38)–(40), we have $\tilde{P}_{\mathbf{s},\mathbf{s}'} = \tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}'}}$ for all $\mathbf{s}'$ in this case.

Case 3. $n(\mathbf{s}) = M$: In this case, we see from (15) that $n(\overline{\mathbf{s}}) = 0$. As such, we have $\tilde{P}_{\overline{\mathbf{s}},\overline{\mathbf{s}'}} = \tilde{P}_{\mathbf{s},\mathbf{s}'}$ for all $\mathbf{s}'$ in this case by following the same argument as in Case 2 with the roles of $\mathbf{s}$ and $\overline{\mathbf{s}}$ interchanged. ■

Lemma 3 *$\tilde{\pi}_{\mathbf{s}} = \tilde{\pi}_{\overline{\mathbf{s}}}$ for all states* $\mathbf{s}$.

Proof. Let $\mathbf{u}$ be the probability vector such that $u_{\mathbf{s}} = \tilde{\pi}_{\overline{\mathbf{s}}}$. From Lemma 2 and $\tilde{\boldsymbol{\pi}} = \tilde{\boldsymbol{\pi}}\tilde{P}$, we can see that

$$\sum_{\mathbf{s}'} u_{\mathbf{s}'} \tilde{P}_{\mathbf{s}',\mathbf{s}} = \sum_{\mathbf{s}'} \tilde{\pi}_{\overline{\mathbf{s}'}} \tilde{P}_{\overline{\mathbf{s}'},\overline{\mathbf{s}}} = \sum_{\mathbf{s}''} \tilde{\pi}_{\mathbf{s}''} \tilde{P}_{\mathbf{s}'',\overline{\mathbf{s}}} = \tilde{\pi}_{\overline{\mathbf{s}}} = u_{\mathbf{s}}.$$

In other words, we have $\mathbf{u} = \mathbf{u}\tilde{P}$. It follows that $\mathbf{u}$ is the unique steady state probability vector $\tilde{\boldsymbol{\pi}}$ of the Markov chain $\{\mathbf{s}(t),\ t \geq 0\}$, i.e., $\tilde{\pi}_{\mathbf{s}} = u_{\mathbf{s}} = \tilde{\pi}_{\overline{\mathbf{s}}}$ for all $\mathbf{s}$. ■

Now we continue the proof of the claim that $T_r = B/M$. Consider a given time slot in steady state, called the tagged time slot. Let $Z_i = 1$ if there is a packet routed into the i^{th} fiber delay line at the tagged time slot and $Z_i = 0$ otherwise for $1 \leq i \leq M$. Let S be the number of packets routed into the M fiber delay lines at the tagged time slot, i.e., $S = \sum_{i=1}^{M} Z_i$.

From Lemma 3 and $r_1 = M$, we have

$$\begin{aligned}P(S = k) &= \sum_{\mathbf{s}: s_1 = k} \tilde{\pi}_{\mathbf{s}} = \sum_{\mathbf{s}: s_1 = k} \tilde{\pi}_{\overline{\mathbf{s}}} = \sum_{\mathbf{s}: s_1 = k} \tilde{\pi}_{\mathbf{r}-\mathbf{s}} \\ &= \sum_{\mathbf{s}: s_1 = r_1 - k} \tilde{\pi}_{\mathbf{s}} = \sum_{\mathbf{s}: s_1 = M - k} \tilde{\pi}_{\mathbf{s}} \\ &= P(S = M - k). \qquad (41)\end{aligned}$$

Since the packets from the shifter's output links are routed to the fiber delay lines "consecutively" starting from the first fiber delay line in our constructions of optical priority queues, we see that $Z_i = 1$ if and only if $S \geq i$. It then follows from (41) that

$$\begin{aligned}&P(Z_i = 1) + P(Z_{M+1-i} = 1) \\ &= \sum_{k=i}^{M} P(S = k) + \sum_{k=M+1-i}^{M} P(S = k) \\ &= \sum_{k=i}^{M} P(S = k) + \sum_{k=M+1-i}^{M} P(S = M - k) \\ &= \sum_{k=i}^{M} P(S = k) + \sum_{k=0}^{i-1} P(S = k) = 1. \qquad (42)\end{aligned}$$

From (41), we can see that the average number of packets routed into the M fiber delay lines at the tagged time slot is given by

$$\sum_{k=0}^{M} k\cdot P(S=k)$$
$$= \begin{cases} \sum_{k=0}^{\frac{M}{2}-1} k\cdot P(S=k) + \frac{M}{2}\cdot P(S=\frac{M}{2}) \\ \quad + \sum_{k=\frac{M}{2}+1}^{M} k\cdot P(S=M-k), \text{ if } M \text{ is even,} \\ \sum_{k=0}^{\lceil M/2\rceil -1} k\cdot P(S=k) \\ \quad + \sum_{k=\lceil M/2\rceil}^{M} k\cdot P(S=M-k), \text{ if } M \text{ is odd,} \end{cases}$$
$$= \begin{cases} \sum_{k=0}^{\frac{M}{2}-1} (k+M-k)\cdot P(S=k) + \frac{M}{2}\cdot P(S=\frac{M}{2}), \\ \qquad\qquad\qquad\qquad \text{if } M \text{ is even,} \\ \sum_{k=0}^{\lceil M/2\rceil -1} (k+M-k)\cdot P(S=k), \\ \qquad\qquad\qquad\qquad \text{if } M \text{ is odd,} \end{cases}$$
$$= \frac{M}{2}\cdot\sum_{k=0}^{M} P(S=k) = \frac{M}{2}. \tag{43}$$

Furthermore, we see from $d_{M+1-i} = d_i$ for $1 \le i \le M$ and (42) that the average recirculation time (per recirculation) through the fiber delay lines of the packets routed into the M fiber delay lines at the tagged time slot is given by

$$\sum_{i=1}^{M} d_i \cdot P(Z_i = 1)$$
$$= \begin{cases} \sum_{i=1}^{M/2} d_i \cdot (P(Z_i=1) + P(Z_{M+1-i}=1)), \\ \qquad\qquad\qquad\qquad \text{if } M \text{ is even,} \\ \sum_{i=1}^{\lceil M/2\rceil -1} d_i \cdot (P(Z_i=1) + P(Z_{M+1-i}=1)) \\ \quad + d_{\lceil M/2\rceil}\cdot P(Z_{\lceil M/2\rceil}=1), \text{ if } M \text{ is odd,} \end{cases}$$
$$= \begin{cases} \sum_{i=1}^{M/2} d_i \cdot 1, \text{ if } M \text{ is even,} \\ \sum_{i=1}^{\lceil M/2\rceil -1} d_i\cdot 1 + d_{\lceil M/2\rceil}\cdot\frac{1}{2}, \text{ if } M \text{ is odd,} \end{cases}$$
$$= \frac{1}{2}\cdot\sum_{i=1}^{M} d_i = \frac{B}{2}. \tag{44}$$

Therefore, the average recirculation time (per recirculation) of an optical packet through the fiber delay lines in steady state can be obtained from (43) and (44) as follows:

$$T_r = \frac{\sum_{i=1}^{M} d_i\cdot P(Z_i=1)}{\sum_{k=0}^{M} k\cdot P(S=k)} = \frac{B/2}{M/2} = \frac{B}{M}.$$

We remark that for the case that $\alpha \neq \beta$, we are not able to obtain T_r in closed form, but we can still compute T_r as follows. First we compute $\tilde{\boldsymbol{\pi}}$ by solving $\tilde{\boldsymbol{\pi}} = \tilde{\boldsymbol{\pi}}\tilde{P}$, where $\tilde{P}$ is given by (19)–(22), (30)–(32), and (35)–(37). Then we calculate $P(S=k) = \sum_{\mathbf{s}:s_1=k}\tilde{\pi}_{\mathbf{s}}$ for $0 \le k \le M$ and $P(Z_i=1) = \sum_{k=i}^{M} P(S=k)$ for $1 \le i \le M$. Finally, T_r can be computed as $T_r = \frac{\sum_{i=1}^{M} d_i\cdot P(Z_i=1)}{\sum_{k=0}^{M} k\cdot P(S=k)}$.

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