Constructions of Optical Priority Queues Under a Priority-Based Routing Policy

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Abstract—In this paper, we give a class of constructions of optical priority queues by using a feedback system consisting of a \((km + 2) \times (kmn + 2)\) (bufferless) optical crossbar switch and \(k\) groups of optical first-in first-out (FIFO) multiplexers, where the \(i^{th}\) group has \(m\) parallel optical \(n\)-to-1 FIFO multiplexers with the same buffer size \(B_i\) for \(i = 1, 2, \ldots, k\). We show that such a feedback system can be operated as an optical priority queue under a simple priority-based routing policy if the conditions in (A1)--(A3) (see Theorem 3) are satisfied. In order to evaluate the effectiveness of SDL constructions of optical priority queues, we introduce the construction efficiency for Switched-Delay-Lines (SDL) constructions of optical priority queues. To maximize the construction efficiency for our constructions, we show that the inequalities in (A1) and (A3) should hold with equality and the buffer sizes \(B_1, B_2, \ldots, B_k\) should satisfy the condition in (B1) (see Section III-A). We also obtain closed-form expressions or bounds for the construction efficiency for the special case that \(B_1, B_2, \ldots, B_k\) are given by (B2). Our results show that our construction is more effective than the best construction in [33] in the literature, and can achieve a buffer size of \(2^{O(\sqrt{M})}\) by using a feedback system consisting of an \((M + 2) \times (M + 2)\) (bufferless) optical crossbar switch and \(M\) fiber delay lines.

Index Terms—Descartes’ rule of signs, Eneström-Kakeya theorem, FIFO multiplexers, optical buffers, optical queues, optical switches, priority queues.

I. INTRODUCTION

One of the challenging problems towards all-optical packet-switching is the design of optical buffers for conflict resolution among packets competing for the same resources. In the last two decades, there have been extensive studies [1]–[51] on the constructions of various types of optical queues by using fiber delay lines as the storage media and using (bufferless) optical crossbar switches to direct optical packets through the fiber delay lines so that the optical packets can be routed to the right places at the right times. Such Switched-Delay-Lines (SDL) constructions of optical queues by using optical crossbar switches and fiber delay lines include the early feasibility studies in [1]–[4], output-buffered switches in [5]–[10], first-in first-out (FIFO) multiplexers in [5] and [10]–[20], FIFO queues in [20]–[25], last-in first-out (LIFO) queues in [22], [23], and [26], priority queues in [27]–[34], time slot interchanges in [20] and [35], linear compressors, linear decompressors, non-overtaking delay lines, and flexible delay lines in [20] and [36]–[41], and FIFO contractors, LIFO contractors, and absolute contractors in [42]. Furthermore, results on the fundamental complexity of SDL constructions of optical queues can be found in [43] and performance analysis for optical queues has been addressed in [44] and [45]. For review articles on SDL constructions of optical queues, we refer to [46]–[51] and the references therein.

In this paper, we focus on the constructions of optical priority queues. The class of optical priority queues is one of the most general types of optical queues and includes optical FIFO queues and optical LIFO queues as special cases. The first construction of an optical priority queue was proposed in [27], in which an optical priority queue with buffer size \(O(M^2)\) was constructed by using a feedback system consisting of an \((M + 2) \times (M + 2)\) (bufferless) optical crossbar switch and \(M\) fiber delay lines. The proof in [27] is quite elaborate, and a simpler proof can be found in [28]. The buffer size \(O(M^2)\) achieved in [27] was improved to \(O(M^3)\) in [31], was improved to \(O(M^c)\) for any positive integer \(c\) in [32], and was improved to \(2^{O(\sqrt{M})}\) in [33].

We note that the construction in [33] is currently the best (in the order sense) construction of an optical priority queue in the literature. Furthermore, the results in [31] have been extended to the scenario of multiple inputs and multiple outputs in [34]. It was shown in [27] that the buffer size of an optical priority queue that is constructed by using SDL elements containing \(M\) fiber delay lines is upper bounded by \(2^M\). This shows that there is a gap between the buffer sizes of all existing constructions of optical priority queues in the literature and the theoretical upper bound \(2^M\). Whether the theoretical upper bound \(2^M\) can be achieved or not, in the case that it can be achieved, how to achieve this upper bound remains a challenging open problem.

In this paper, we adopt a generalization of the architecture and the priority-based routing policy proposed in one of the authors’ previous work [33], and obtain a class of constructions of optical priority queues. In our constructions, we use the feedback system in Figure 1 consisting of a \((km + 2) \times (kmn + 2)\) (bufferless) optical crossbar switch and \(k\) groups of optical FIFO multiplexers, where the \(i^{th}\) group has \(m\) parallel optical \(n\)-to-1 FIFO multiplexers with the same buffer size \(B_i\) for \(i = 1, 2, \ldots, k\). We note that the feedback system in [33] is a special of that in Figure 1 when \(k = 2\ell - 1\) for some \(\ell \geq 2, m = 3, n = 4\). Suppose that \(1 \leq s \leq k - 1\) and \(m \geq 1\), where \(s\) is a parameter used in the description of the priority-based routing policy described in (R1)–(R3) (see Section II-B). We note that the priority-based routing policy proposed in [33] is a special case of that in (R1)–(R3) when
s = 1. Then we show in Theorem 3 (see Section II-C) that the feedback system in Figure 1 can be operated as an optical priority queue under the priority-based routing policy in (R1)–(R3) if the conditions in (A1)–(A3) in Theorem 3 are satisfied. 

In order to evaluate the effectiveness of SDL constructions of optical priority queues, we introduce the construction efficiency for SDL constructions of optical priority queues in Section III-A. We show that to maximize the construction efficiency for our constructions, the inequalities in (A1) and (A3) should hold with equality and the buffer sizes \( B_1, B_2, \ldots, B_k \) should satisfy the condition in (B1) (see Section III-A). We denote \( \rho_{s,k,m}(B) \) as the construction efficiency for such a construction, where \( B = (B_1, B_2, \ldots, B_k) \). It turns out that it is difficult to obtain analytical results on the maximum construction efficiency \( \rho^*_{s,k,m} \) of \( \rho_{s,k,m}(B) \) over all choices of \( B_1, B_2, \ldots, B_k \) satisfying the condition in (B1) due to the combinatorial nature and the discontinuity of the objective function of such an optimization problem. From our numerical results, we observe that the construction efficiency \( \tilde{\rho}_{s,k,m} \) for the special case that \( B_1, B_2, \ldots, B_k \) are given by (B2) (see Section III-A) is close to the maximum construction efficiency \( \rho^*_{s,k,m} \) in many scenarios. As such, in Section III-B we first obtain closed-form expressions for \( B_1, B_2, \ldots, B_k \) that are given by (B2), and then obtain closed-form expressions or bounds for \( \rho_{s,k,m} \). The results on \( \rho_{s,k,m} \) show that \( \rho_{s,k,m} \) decreases as \( s, k, \) or \( m \) increases, and show that our construction is more effective than the best construction in [33] in the literature. Furthermore, the results on \( \tilde{\rho}_{s,k,m} \) also show that if our construction is implemented by using a feedback system consisting of an \((M+2) \times (M+2)\) (bufferless) optical crossbar switch and \( M \) fiber delay lines, then the buffer size achieved is \( \Theta(M^c) \) for some 1 ≤ c ≤ 2 or \( 2^\Theta(\sqrt{M}) \).

The rest of this paper is organized as follows. In Section II, we give the formal definitions of priority queues and \( n\text{-to-1 FIFO multiplexers} \). FIFO multiplexers, describe our priority-based routing policy performed by the crossbar switch in Figure 1, and show that the feedback system in Figure 1 can be operated as an optical priority queue under our priority-based routing policy if the conditions in (A1)–(A3) are satisfied. In Section III, we introduce the construction efficiency for SDL constructions of optical priority queues, show that the inequalities in (A1) and (A3) should hold with equality and the buffer sizes \( B_1, B_2, \ldots, B_k \) should satisfy the condition in (B1) in order to maximize the construction efficiency for our constructions, and obtain closed-form expressions or bounds for the construction efficiency for the special case that \( B_1, B_2, \ldots, B_k \) are given by (B2). Finally, we conclude this paper in Section IV.

II. CONSTRUCTIONS OF OPTICAL PRIORITY QUEUES

We first give the formal definitions of priority queues and \( n\text{-to-1 FIFO multiplexers} \) in Section II-A. Then in Section II-B we describe the priority-based routing policy performed by the crossbar switch in Figure 1. Finally, in Section II-C we show that the feedback system in Figure 1 can be operated as an optical priority queue under our priority-based routing policy if the conditions in (A1)–(A3) are satisfied. As in most works on SDL constructions of optical queues in the literature, in this paper we consider the discrete-time setting in which time is slotted and synchronized, and we assume that packets are of the same size so that a packet can be transmitted within a time slot. We note that variable-size packets can be easily taken care of by introducing packet segmentation at the source and packet reassembly at the destination.

A. Priority Queues and \( n\text{-to-1 FIFO Multiplexers} \)

**Definition 1 (Priority Queues)** A priority queue with buffer size \( B \) is a network element with one arrival link, one departure link, one loss link, and one control input (see Figure 2). We denote \( a(t) = 1 \) (resp., \( a(t) = 0 \)) if there is a packet (resp., there is no packet) arriving from the arrival link at time \( t \); we denote \( d(t) = 1 \) (resp., \( d(t) = 0 \)) if there is a packet (resp., there is no packet) departing from the departure link at time \( t \); and we denote \( l(t) = 1 \) (resp., \( l(t) = 0 \)) if there is a packet (resp., there is no packet) dumped through the loss link at time \( t \). We denote \( c(t) = 1 \) (resp., \( c(t) = 0 \)) if there is a departure request (resp., there is no departure request) at time \( t \). Let \( q(t) \) be the number of packets stored in the buffer at time \( t \) (at the end of the \( t \)-th time slot). We assume that all of the packets in the queue at time \( t \) (at the beginning of the \( t \)-th time slot), including the \( q(t-1) \) packets stored in the buffer at time \( t - 1 \)
and the \( a(t) \) packets arriving from the arrival link at time \( t \), have distinct priorities. At time \( t \) (at the beginning of the \( t \)-th time slot), we number the priorities of the \( q(t-1) + a(t) \) packets in the queue at time \( t \) from 1 (the highest priority) to \( q(t-1) + a(t) \) (the lowest priority), and we denote the priority of a packet \( p \) in the queue at time \( t \) as \( \tau_p(t) \). Then a priority queue with buffer size \( B \) satisfies the following five properties:

(P1) Flow conservation: Packets from the arrival link are either stored in the buffer or transmitted through the departure link or the loss link. Therefore, we have

\[ q(t) = q(t-1) + a(t) - d(t) - \ell(t). \]  

(1)

(P2) Nonidling: If there is a departure request at time \( t \) and there are packets in the queue at time \( t \), then there is a departure packet at time \( t \); otherwise, there is no departure packet at time \( t \). Therefore, we have

\[ d(t) = \begin{cases} 1, & \text{if } c(t) = 1 \text{ and } q(t-1) + a(t) > 0, \\ 0, & \text{otherwise}. \end{cases} \]  

(2)

(P3) Maximum buffer usage: If there is a buffer overflow at time \( t \), i.e., if there is no departure request at time \( t \), the buffer is full at time \( t-1 \), and there is an arrival packet at time \( t \), then there is a loss packet at time \( t \); otherwise, there is no loss packet at time \( t \). Therefore, we have

\[ \ell(t) = \begin{cases} 1, & \text{if } c(t) = 0, \text{ } q(t-1) = B, \text{ and } a(t) = 1, \\ 0, & \text{otherwise}. \end{cases} \]  

(3)

(P4) Priority departure: If there is a departure packet, say packet \( p \), at time \( t \), then packet \( p \) is the highest priority packet in the queue at time \( t \), i.e., \( \tau_p(t) = 1 \).

(P5) Priority loss: If there is a loss packet, say packet \( p \), at time \( t \), then packet \( p \) is the lowest priority packet in the queue at time \( t \), i.e., \( \tau_p(t) = B + 1 \).

Let \( a(t) = \sum_{i=1}^{n} a_i(t) \) (resp., \( \ell(t) = \sum_{i=1}^{n-1} \ell_i(t) \)) be the total number of packets arriving from the \( n \) arrival links (resp., dumped through the \( n-1 \) loss links) at time \( t \). Also let \( q(t) \) be the number of packets stored in the buffer at time \( t \) (at the end of the \( t \)-th time slot). We assume that the arrival links are prioritized so that the priorities of the arrival links are decreasing in the link indices, i.e., among packets arriving at the same time, packets from arrival links with smaller link indices are regarded as arriving earlier than those from arrival links with larger link indices. Then an \( n \)-to-1 FIFO multiplexer with buffer size \( B \) satisfies the following five properties:

(M1) Flow conservation: Packets from the \( n \) arrival links are either stored in the buffer or transmitted through the departure link or the \( n-1 \) loss links. Therefore, we have

\[ q(t) = q(t-1) + a(t) - d(t) - \ell(t). \]  

(4)

(M2) Nonidling: If there are packets in the queue at time \( t \), then there is a departure packet at time \( t \); otherwise, there is no departure packet at time \( t \). Therefore, we have

\[ d(t) = \begin{cases} 1, & \text{if } q(t-1) + a(t) > 0, \\ 0, & \text{otherwise}. \end{cases} \]  

(5)

(M3) Maximum buffer usage: If there is a buffer overflow at time \( t \), i.e., if \( q(t-1) + a(t) - 1 > B \), then there are \( q(t-1) + a(t) - 1 - B \) loss packets at time \( t \); otherwise, there are no loss packets at time \( t \). Therefore, we have

\[ \ell(t) = \begin{cases} q(t-1) + a(t) - 1 - B, & \text{if } q(t-1) + a(t) - 1 > B, \\ 0, & \text{otherwise}. \end{cases} \]  

(6)

(M4) FIFO departure: Packets depart in the FIFO order.

(M5) FIFO loss with prioritized loss links: If there are loss packets at time \( t \), i.e., \( \ell(t) > 0 \) (note that from (6) we have \( \ell(t) = q(t-1) + a(t) - 1 - B \)), then the loss packets are the last \( \ell(t) \) arrival packets at time \( t \), and they are dumped through loss links \( 1, 2, \ldots, \ell(t) \) in the order of increasing arrival link indices, i.e., the packet arriving from arrival link \( a(t) - \ell(t) + i \) at time \( t \) is dumped through loss link \( i \) at time \( t \) for \( i = 1, 2, \ldots, \ell(t) \). Therefore, we have

\[ \ell_i(t) = \begin{cases} 1, & \text{if } q(t-1) + a(t) - 1 - B \geq i, \\ 0, & \text{otherwise}, \end{cases} \]  

for \( i = 1, 2, \ldots, n-1 \).

\[ \text{B. The Priority-Based Routing Policy} \]

In the rest of this paper, the notations \( a(t), d(t), \ell(t), c(t), \) and \( q(t) \) are as defined in Definition 1 for a priority queue for all \( t \geq 0 \).

To describe the priority-based routing policy in this paper, we let \( U_0 = 0 \) and let \( \Psi_i = \{L_i, L_i + 1, \ldots, U_i\} \) be a set consisting of consecutive integers \( L_i, L_i + 1, \ldots, U_i \) for \( i = 1, 2, \ldots, k \), where

\[ L_i \leq U_i \text{ and } L_i = U_{i-1} + 1 \text{ for } i = 1, 2, \ldots, k. \]  

(8)
It is clear from (8) that $L_1 = U_0 + 1 = 1$ and the only unknown variables are $U_1, U_2, \ldots, U_k$ that satisfy
\[ 1 \leq U_1 < U_2 < \cdots < U_k. \] (9)
Furthermore, it is easy to see that the sets $\Psi_1, \Psi_2, \ldots, \Psi_k$ are pairwise disjoint and $\bigcup_{j=1}^k \Psi_j = \{1, 2, \ldots, U_k\}$ is a set consisting of consecutive integers $1, 2, \ldots, U_i$ for $i = 1, 2, \ldots, k$. Therefore, we have
\[ U_i = \sum_{j=1}^i |\Psi_j| \text{ for } i = 1, 2, \ldots, k. \] (10)

Suppose that $1 \leq s \leq k - 1$. Then we operate the crossbar switch in Figure 1 according to the following priority-based routing policy in (R1)–(R3) at all times $t$ (at the beginning of the $t^{th}$ time slot) for all $t \geq 0$:

(R1) If there is a departure request at time $t$, i.e., $c(t) = 1$, then the highest priority packet, if any, among all of the packets from the arrival link or the $m(s + 1)$ output links of the first $s + 1$ groups of FIFO multiplexers at time $t$ is routed to the departure link at time $t$.

(R2) If there is a buffer overflow at time $t$, i.e., $c(t) = 0$, then the lowest priority packet, if any, among all of the packets from the arrival link or the $m$ output links of the last group of FIFO multiplexers at time $t$ is routed to the loss link at time $t$.

(R3) The other packets, i.e., the packets not routed to the departure link according to (R1) or the loss link according to (R2), at the input links of the crossbar switch in Figure 1 at time $t$ with priorities belonging to the set $\Psi_j$, are routed to the $j^{th}$ group of FIFO multiplexers at time $t$ in a round-robin fashion so that the $m$ FIFO multiplexers in the $j^{th}$ group are evenly loaded for all $i = 1, 2, \ldots, k$. Specifically, the round-robin routing is similar to the join-the-shortest-queue policy in queueing theory and can be described as follows. Consider the $i^{th}$ group, where $1 \leq i \leq k$. We call arrival link $\ell$ of the $j^{th}$ FIFO multiplexer in the $i^{th}$ group the $(\ell - 1)m + j - 1)^{th}$ input link of the $i^{th}$ group for $j = 1, 2, \ldots, m$ and $\ell = 1, 2, \ldots, n$. For $t \geq 0$, we define $u(t) = mn - 1$ if there are no packets routed to the $i^{th}$ group before or at time $t$; otherwise, we denote $u(t)$ as the index of the input link of the $i^{th}$ group that is lastly used before or at time $t$. If there are $r_i(t)$ packets routed to the $i^{th}$ group at time $t$, then they are routed to the $((u(t) + 1) + 1) \mod mn$th, $(u(t) + 2) \mod mn \mod mn \cdots, ((u(t) + 1) + r_i(t)) \mod mn$th input links of the $i^{th}$ group, and we have $u(t) = (u(t - 1) + r_i(t)) \mod mn$.

C. The Constructions of Optical Priority Queues

In the following theorem, we show that the feedback system in Figure 1 can be operated as an optical priority queue with buffer size $U_k$ under the priority-based routing policy in (R1)–(R3) if the conditions in (A1)–(A3) are satisfied.

**Theorem 3** Suppose that $1 \leq s \leq k - 1$ and $m \geq 1$, and suppose that $n$, $B_1, B_2, \ldots, B_k$, $|\Psi_1|$, $|\Psi_2|, \ldots, |\Psi_k|$ satisfy the following conditions in (A1)–(A3):

(A1) $n \geq \min\{2s + 1, k\} + 1$.

(A2) $B_1 = B_k = 1$.

\[ 1 \leq B_i \leq \begin{cases} U_{i-1}, & \text{if } 2 \leq i \leq s, \\ U_{i-1} - U_{i-s-1}, & \text{if } s + 1 \leq i \leq k, \\ U_{i+s} - U_{i}, & \text{if } 1 \leq i \leq k - s, \\ U_k - U_{k-s+1}, & \text{if } k - s + 1 \leq i \leq k. \end{cases} \]

(A3) $|\Psi_i| \leq (m - 1)B_i + 1$ for $i = 1, 2, \ldots, k$.

Assume that the feedback system in Figure 1 is started from an empty system at time 0 and is operated under the priority-based routing policy in (R1)–(R3) at all times $t \geq 0$. Then the routing policy in (R1)–(R3) is feasible and the feedback system in Figure 1 can be operated as an optical priority queue with buffer size $U_k$ at all times $t \geq 0$.

**Remark 4** We note that the construction in [33] is a special case of that in Theorem 3 when $s = 1$, $k = 2\ell - 1$ for some $\ell \geq 2$, $m = 3$, $n = 4$.

\[ B_i = B_{k-i+1} = \begin{cases} 1, & \text{if } i = 1, \\ 2i-2, & \text{if } 2 \leq i \leq \ell, \end{cases} \] (11)

and $|\Psi_i| = |\Psi_{k-i+1}| = \begin{cases} 1, & \text{if } i = 1, \\ 2i-1, & \text{if } 2 \leq i \leq \ell. \end{cases}$ (12)

We need the following three lemmas from [33] for the proof of Theorem 3. In these lemmas, we assume that the feedback system in Figure 1 is operated as an optical priority queue with buffer size $U_k$ up to time $t - 1$ for some $t \geq 1$.

The first lemma says that the priority of a packet $p$ in the queue can only change by at most one in a time slot, which is a direct result of the fact that there is at most one arrival packet and there is at most one departure packet with priorities higher than that of packet $p$ in a time slot.

**Lemma 5** [33, Lemma 5] Assume that the feedback system in Figure 1 is operated as an optical priority queue with buffer size $U_k$ up to time $t - 1$ for some $t \geq 1$. Suppose that a packet $p$ is buffered in the feedback system in Figure 1 at time $t' - 1$ for some $t' \leq t$. Then we have $|\tau_p(t') - \tau_p(t' - 1)| \leq 1$.

The second lemma gives the range of the priority of a packet buffered at some group of FIFO multiplexers under the routing policy in (R1)–(R3).

**Lemma 6** [33, Lemma 7] Assume that the feedback system in Figure 1 is operated under the routing policy in (R1)–(R3) at all times and is operated as an optical priority queue with buffer size $U_k$ up to time $t - 1$ for some $t \geq 1$. Suppose that a
packet \( p \) is buffered at the \( i \)^th group of FIFO multiplexers at time \( t' \) for some \( 1 \leq i \leq k \) and \( t' \leq t \). Then we have

\[ I_i - B_i + 1 \leq \tau_p(t') \leq U_i + B_i - 1. \]

The third lemma gives an upper bound on the number of packets buffered at or routed to some group of FIFO multiplexers at time \( t \) under the routing policy in (R1)–(R3).

**Lemma 7** [33, Lemma 10] Assume that the feedback system in Figure 1 is operated under the routing policy in (R1)–(R3) at all times and is operated as an optical priority queue with buffer size \( U_k \) up to time \( t-1 \) for some \( t \geq 1 \). Then there are at most \( |W_i| + B_i - 1 \) packets buffered at or routed to the \( i \)^th group of FIFO multiplexer at time \( t \).

We will prove Theorem 3 by induction on time \( t \). In the proof, we assume as the induction hypothesis that the routing policy in (R1)–(R3) is feasible and the feedback system in Figure 1 is operated as an optical priority queue with buffer size \( U_k \) up to time \( t-1 \) for some \( t \geq 1 \). We will use Lemma 5, Lemma 6, (A1), and (A2) to show that there is no collision at any input link of any FIFO multiplexer at time \( t \). Thus, the routing policy in (R1)–(R3) is feasible at time \( t \). We will use Lemma 7 and (A3) to show that there is no packet loss at any FIFO multiplexer at time \( t \). As there is no collision at any input link of any FIFO multiplexer and there is no packet loss at any FIFO multiplexer at time \( t \), it is clear that there is no internal loss in the feedback system in Figure 1 at time \( t \). Thus, the flow conservation property in (P1) (with \( B = U_k \)) is satisfied at time \( t \). We will also use Lemma 5, Lemma 6, (A1), and (A2) to show that the properties in (P2)–(P5) (with \( B = U_k \)) are satisfied at time \( t \). Therefore, the induction is completed.

**Proof.** (Proof of Theorem 3) We will prove Theorem 3 by induction on time \( t \). Since we have assumed that the feedback system in Figure 1 is started from an empty system at time 0, the routing policy in (R1)–(R3) is feasible and the properties in (P1)–(P5) in Definition 1 (with \( B = U_k \)) are satisfied at time 0. Assume as the induction hypothesis that the routing policy in (R1)–(R3) is feasible and the properties in (P1)–(P5) in Definition 1 (with \( B = U_k \)) are satisfied up to time \( t-1 \) for some \( t \geq 1 \). In the following, we show that the routing policy in (R1)–(R3) is feasible and the properties in (P1)–(P5) in Definition 1 (with \( B = U_k \)) are satisfied at time \( t \), and the induction is completed.

(i) The routing policy in (R1)–(R3) is feasible at time \( t \). We will use Lemma 5, Lemma 6, and (A2) to show that there are at most \( m \cdot \min\{2s+1, k\} + 1 \) packets routed to any group of FIFO multiplexers at time \( t \). As there are \( mn \) input links at any group of FIFO multiplexers and we can see from (A1) and \( m \geq 1 \) that \( mn \geq m \cdot \min\{2s+1, k\} + 1 \), it then follows from the routing policy in (R3) that there is at most one packet routed to any input link of any FIFO multiplexer at time \( t \), and hence there is no collision at any input link of any FIFO multiplexer at time \( t \). Thus, the routing policy in (R1)–(R3) is feasible at time \( t \).

To show that there are at most \( m \cdot \min\{2s+1, k\} + 1 \) packets routed to any group of FIFO multiplexers at time \( t \), consider a packet, say packet \( p \), that is buffered at the \( i \)^th group of FIFO multiplexers at time \( t-1 \) and leaves the \( j \)^th group of FIFO multiplexers and appears at one of the input links of the crossbar switch in Figure 1 at time \( t \), where \( 1 \leq i \leq k \). If \( s + 1 \leq i \leq k \), then we have

\[ \tau_p(t) - L_{i-s} \geq (\tau_p(t-1) - L_{i-s}) - (L_i - B_i) - L_{i-s} = (U_{i-1} + B_i) - (U_{i-s-1} + 1) \geq 0, \]

where the first inequality follows from Lemma 5, the second inequality follows from Lemma 6, the equality follows from (8), and the third inequality follows from (A2). Similarly, if \( 1 \leq i \leq k - s \), then we also have from Lemma 5, Lemma 6, and (A2) that

\[ \tau_p(t) - U_{i+s} \leq (\tau_p(t-1) + 1) - U_{i+s} \leq (U_i + B_i) - U_{i+s} \leq 0. \]

Thus, we can see from the routing policy in (R1)–(R3), (13), and (14) that packet \( p \) can only be routed to the departure link, the loss link, or the \((i-s)^{th}\) group (if \( s + 1 \leq i \leq k \)), the \((i-s+1)^{th}\) group (if \( s \leq i \leq k \)),..., the \( i^{th} \) group (if \( 1 \leq i \leq k \)),..., the \((i+s-1)^{th}\) group (if \( 1 \leq i \leq k - s + 1 \)), or the \((i+s)^{th}\) group (if \( 1 \leq i \leq k - s \)) of FIFO multiplexers at time \( t \).

Therefore, we deduce that the packets routed to the \( i^{th} \) group of FIFO multiplexers at time \( t \) can only come from the arrival link or the output links of the \((i-s)^{th}\) group (if \( s + 1 \leq i \leq k \)), the \((i-s+1)^{th}\) group (if \( s \leq i \leq k \)),..., the \( i^{th} \) group (if \( 1 \leq i \leq k \)),..., the \((i+s-1)^{th}\) group (if \( 1 \leq i \leq k - s + 1 \)), or the \((i+s)^{th}\) group (if \( 1 \leq i \leq k - s \)) of FIFO multiplexers. As each group has \( m \) n-to-1 FIFO multiplexers, we conclude that there are at most \( m \cdot \min\{2s+1, k\} + 1 \) packets routed to the \( i^{th} \) group of FIFO multiplexers at time \( t \).

(ii) The property in (P1) in Definition 1 (with \( B = U_k \)) is satisfied at time \( t \). We will use Lemma 7 and (A3) to show that there is no packet loss at any FIFO multiplexer at time \( t \). As we know from (i) that there is no collision at any input link of any FIFO multiplexer at time \( t \), it is then clear that there is no internal loss in the feedback system in Figure 1 at time \( t \). Thus, the flow conservation property in (P1) is satisfied at time \( t \).

To show that there is no packet loss at any FIFO multiplexer at time \( t \), consider the \( i^{th} \) group of FIFO multiplexers, where \( 1 \leq i \leq k \). Let \( q_{i,j}(t') \) (resp., \( a_{i,j}(t') \)) be the number of packets stored in the buffer of (resp., the number of packets routed to) the \( j^{th} \) FIFO multiplexer in the \( j^{th} \) group at time \( t' \) for \( j = 1, 2, \ldots, m \) and \( t' = 0, 1, 2, \ldots \). Since the \( m \) FIFO multiplexers in the \( j^{th} \) group are evenly loaded (according to the routing policy in (R3)) and evenly served (according to the nonidling property in (M2)), and there is no packet loss at any FIFO multiplexer up to time \( t - 1 \) (according to the induction hypothesis), it is a direct result of the join-the-shortest-queue and serve-the-longest-queue policy in queueing theory that the \( m \) numbers \( q_{i,j}(t+1) + a_{i,j}(t) \), \( j = 1, 2, \ldots, m \), differ by at most one, i.e.,

\[ |(q_{i,j}(t+1) + a_{i,j}(t) - 1) - (q_{i,j}(t) + a_{i,j}(t) - 1)| \leq 1 \text{ for all } 1 \leq j, j' \leq m. \]
According to the nonidling property in (M2), the number of packets buffered at or routed to the $j^\text{th}$ FIFO multiplexer in the $i^\text{th}$ group at time $t$ is given by $(q_{i,j}(t-1) - 1)^+ + a_{i,j}(t)$ for $j = 1, 2, \ldots, m$. Thus, we have

$$
\sum_{j=1}^{m} (q_{i,j}(t-1) + a_{i,j}(t-1))^+ \\
\leq \sum_{j=1}^{m} ((q_{i,j}(t-1) - 1)^+ + a_{i,j}(t)) \\
\leq |\Psi_i| + B_i - 1 \leq mB_i,
$$

(16)

where the second inequality follows from Lemma 7 and the third inequality follows from (A3). It then follows from (15) and (16) that

$$(q_{i,j}(t-1) + a_{i,j}(t-1))^+ \leq \lceil (mB_i)/m \rceil = B_i
$$

(17)

for $j = 1, 2, \ldots, m$. Therefore, we see from the maximum buffer usage property in (M3) and (17) that there is no packet loss at the $j^\text{th}$ FIFO multiplexer in the $j^\text{th}$ group at time $t$ for all $j = 1, 2, \ldots, m$.

(iii) The properties in (P2) and (P4) in Definition 1 (with $B = U_k$) are satisfied at time $t$. We consider the following two cases.

Case 1: $c(t) = 0$ or $q(t-1) + a(t) = 0$. In this case, there is no departure packet at time $t$ according to the routing policy in (R1)–(R3). Thus, the properties in (P2) and (P4) are satisfied at time $t$.

Case 2: $c(t) = 1$ and $q(t-1) + a(t) > 0$. Let packet $p$ be the highest priority packet in the queue at time $t$, i.e., $\tau_p(t) = 1$. We discuss the following two subcases.

Subcase 2(a): Packet $p$ is an arrival packet at time $t$. In this subcase, packet $p$ is routed to the departure link at time $t$ according to the routing policy in (R1). Thus, the properties in (P2) and (P4) are satisfied at time $t$.

Subcase 2(b): Packet $p$ is not an arrival packet at time $t$. In this subcase, packet $p$ must be stored in the buffer of the queue at time $t-1$. It is clear from Lemma 5, $\tau_p(t-1) = 1$, and

$$
\tau_p(t-1) = 1 \text{ or } \tau_p(t-1) = 2.
$$

(18)

From $L_1 = 1$ in (8) and $B_1 = 1$ in (A2), we have

$$
L_1 - B_1 + 1 = 1 \text{ and } U_1 + B_1 - 1 = U_1.
$$

(19)

From (8), (A2) (note that $U_0 = 0$), and (9), we also have

$$
L_i - B_i + 1 = (U_{i-1} + 1) - B_i + 1 \\
\geq \begin{cases} 
2, & \text{if } 2 \leq i \leq s+1, \\
U_{i-1} + 2 \geq (i - s - 1) + 2 \geq 3, & \text{if } s+2 \leq i \leq k,
\end{cases}
$$

(20)

where, for $2 \leq i \leq s + 1$, the inequality holds with equality if and only if $B_i = U_{i-1}$. Therefore, we can see from Lemma 6 and (18)–(20) that if $\tau_p(t-1) = 1$, then packet $p$ must be buffered at the first group of FIFO multiplexers at time $t-1$, and if $\tau_p(t-1) = 2$, then packet $p$ must be buffered at the first group of FIFO multiplexers at time $t-1$ (in this case we must have $U_1 \geq 2$) or the $i^\text{th}$ group of FIFO multiplexers at time $t-1$ for some $2 \leq i \leq s + 1$ (in this case we must have $B_i = U_{i-1}$).

If packet $p$ is buffered at the first group of FIFO multiplexers at time $t-1$, then it follows from $B_1 = 1$ in (A2) that packet $p$ must depart from the first group of FIFO multiplexers and appear at one of the input links of the crossbar switch in Figure 1 at time $t$, and hence packet $p$ is routed to the departure link at time $t$ according to the routing policy in (R1). Thus, the properties in (P2) and (P4) are satisfied at time $t$.

On the other hand, if packet $p$ is buffered at the $j^\text{th}$ group of FIFO multiplexers at time $t-1$ for some $2 \leq i \leq s+1$, then it must be the case that $\tau_p(t-1) = 2$ and $B_i = U_{i-1}$. We claim that packet $p$ must depart from the $i^\text{th}$ group of FIFO multiplexers and appear at one of the input links of the crossbar switch in Figure 1 at time $t$, and hence packet $p$ is routed to the departure link at time $t$ according to the routing policy in (R1). Thus, the properties in (P2) and (P4) are satisfied at time $t$.

To prove the claim, let $t'$ be the time that packet $p$ is routed to the $i^\text{th}$ group of FIFO multiplexers for the last time before or at time $t-1$. Then we have

$$
B_i = U_{i-1} = L_i - 1 \leq \tau_p(t') - \tau_p(t-1) + 1 \\
= \sum_{\ell=1}^{t-t'-1} (\tau_p(t' + \ell - 1) - \tau_p(t' + \ell)) + 1 \\
\leq (t - t' - 1) + 1 = t - t',
$$

(21)

where the second equality follows from (8), the first inequality follows from $\tau_p(t') \in \Psi_i = \{L_i, L_i+1, \ldots, U_i\}$ (according to the routing policy in (R3)) and $\tau_p(t-1) = 2$, and the second inequality follows from Lemma 5. It then follows from the nonidling property in (M2), the FIFO departure property in (M4), and $t \geq t' + B_i$ in (21) that packet $p$ must depart from the $i^\text{th}$ group of FIFO multiplexers at or before time $t$. As packet $p$ is buffered at the $i^\text{th}$ group of FIFO multiplexers at time $t-1$, it is clear from the definition of $t'$ that packet $p$ is admitted into the $i^\text{th}$ group of FIFO multiplexers at time $t'$, then is buffered at the $i^\text{th}$ group of FIFO multiplexers at times $t', t'+1, \ldots, t-1$, and finally departs from the $i^\text{th}$ group of FIFO multiplexers and appears at one of the input links of the crossbar switch in Figure 1 at time $t$.

(iv) The properties in (P2) and (P5) in Definition 1 (with $B = U_k$) are satisfied at time $t$. We consider the following two cases.

Case 1: $c(t) = 1$ or $q(t-1) + a(t) < U_k + 1$. In this case, there is no loss packet at time $t$ according to the routing policy in (R1)–(R3). Thus, the properties in (P3) and (P5) are satisfied at time $t$.

Case 2: $c(t) = 0$ and $q(t-1) + a(t) \geq U_k + 1$. Since we have from the induction hypothesis that the properties in (P1)–(P5) are satisfied up to time $t-1$, it is easy to see that

$$
g(t') = \min\{q(t' - 1) + a(t') - c(t')\} + U_k \leq U_k
$$

(22)

for $t' = 1, 2, \ldots, t-1$. Thus, it is clear from (22) that in this case we must have $g(t-1) = U_k$ and $a(t) = 1$. Let packet $p$ be the lowest priority packet in the queue at time $t$, i.e., $\tau_p(t) = g(t-1) + a(t) = U_k + 1$. We discuss the following two subcases.
Subcase 2(a): Packet p is the arrival packet at time t. In this subcase, packet p is routed to the loss link at time t according to the routing policy in (R2). Thus, the properties in (P3) and (P5) are satisfied at time t.

Subcase 2(b): Packet p is not the arrival packet at time t. In this subcase, packet p must be stored in the buffer of the queue at time t − 1. It is clear from Lemma 5, τ_p(t) = U_k + 1, and τ_p(t − 1) ≤ q(t − 2) + a(t − 1) ≤ U_k + 1 by (22) that

\[
\tau_p(t - 1) = U_k \quad \text{or} \quad \tau_p(t - 1) = U_k + 1.
\]

(23)

From B_k = 1 in (A2), we have

\[
L_k - B_k + 1 = L_k \quad \text{and} \quad U_k + B_k - 1 = U_k.
\]

(24)

From (A2) and (9), we also have

\[
U_i + B_i - 1 \leq \begin{cases} 
U_{i+s} - 1 \leq U_k - 1, & \text{if } 1 \leq i \leq k - s, \\
U_k - 1, & \text{if } k - s + 1 \leq i \leq k - 1.
\end{cases}
\]

(25)

Therefore, we can see from Lemma 6 and (23)–(25) that it must be the case that \( \tau_p(t - 1) = U_k \) and packet p must be buffered at the last group, i.e., the k-th group, of FIFO multiplexers at time t − 1. As such, it follows from B_k = 1 that packet p must appear at one of the input links of the switch at time t, and hence packet p is routed to the loss link at time t according to the routing policy in (R2). Thus, the properties in (P3) and (P5) are satisfied at time t.

III. CONSTRUCTION EFFICIENCY FOR SDL CONSTRUCTIONS OF OPTICAL PRIORITY QUEUES

In Section III-A, we first introduce the construction efficiency as a performance measure for SDL constructions of optical priority queues, and then show that the inequalities in (A1) and (A3) should hold with equality and the buffer sizes \( B_1, B_2, \ldots, B_k \) should satisfy the condition in (B1) below in order to maximize the construction efficiency for our constructions. As it is difficult to obtain analytical results on the maximum construction efficiency \( \rho_{s,k,m}^* \) over all choices of \( B_1, B_2, \ldots, B_k \) satisfying the condition in (B1) and our numerical results show that the construction efficiency \( \rho_{s,k,m} \) for the special case that \( B_1, B_2, \ldots, B_k \) are given by (B2) below is close to the maximum construction efficiency \( \rho_{s,k,m}^* \) in many scenarios, we obtain closed-form expressions or bounds for \( \rho_{s,k,m} \) in Section III-B.

A. The Construction Efficiency

In order to compare different SDL constructions of optical priority queues, we introduce the construction efficiency as a performance measure for SDL constructions of optical priority queues as follows. Suppose that an optical priority queue with buffer size \( B \) is constructed by using SDL elements containing M fiber delay lines. Then we define the construction efficiency \( \rho \) for such a construction as

\[
\rho = \frac{\log_2 B}{M}.
\]

(26)

As we know from Section I that the buffer size \( B \) for such a construction is upper bounded by \( 2^M \), we have \( 0 \leq \rho \leq 1 \).

For SDL constructions of optical priority queues using an \( (M + 2) \times (M + 2) \) (bufferless) optical crossbar switch and M fiber delay lines, those achieving larger buffer sizes have larger construction efficiencies than the others. Therefore, constructions with larger construction efficiencies are considered more effective than those with smaller construction efficiencies.

Suppose that \( 1 \leq s \leq k - 1 \) and \( m \geq 1 \), and suppose that the conditions in (A1)–(A3) are satisfied. Then we know from Theorem 3 that the feedback system in Figure 1 can be operated as an optical priority queue with buffer size \( U_k \) under the routing policy in (R1)–(R3).

III-B. Construction Efficiency for SDL Constructions of Optical Priority Queues

In Section III-B, we first introduce the construction efficiency as a performance measure for SDL constructions of optical priority queues, and then show that the inequalities in (A1) and (A3) should hold with equality and the buffer sizes \( B_1, B_2, \ldots, B_k \) should satisfy the condition in (B1) below in order to maximize the construction efficiency for our constructions. As it is difficult to obtain analytical results on the maximum construction efficiency \( \rho_{s,k,m}^* \) over all choices of \( B_1, B_2, \ldots, B_k \) satisfying the condition in (B1) and our numerical results show that the construction efficiency \( \rho_{s,k,m} \) for the special case that \( B_1, B_2, \ldots, B_k \) are given by (B2) below is close to the maximum construction efficiency \( \rho_{s,k,m}^* \) in many scenarios, we obtain closed-form expressions or bounds for \( \rho_{s,k,m} \) in Section III-B.

A. The Construction Efficiency

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As we know from Section I that the buffer size \( B \) for such a construction is upper bounded by \( 2^M \), we have \( 0 \leq \rho \leq 1 \).
consisting of an \((\ell(n-1)+n) \times (\ell(n-1)+n)\) (bufferless) optical crossbar switch and \((n-1)\) fiber delay lines. As such, it is easy to see that an optical \(n\)-to-1 FIFO multiplexer with buffer size \(B\) can be constructed by using the feedback system in Figure 5 with \(\ell = \lceil \log_m(B+1) \rceil\).

Therefore, we conclude from the above arguments that an optical priority queue with buffer size \(U_k\) can be constructed by using a feedback system consisting of an \((M+2)\times(M+2)\) (bufferless) optical crossbar switch and \(M\) fiber delay lines, where \(M = m \sum_{i=1}^{k} ((n-1)\lceil \log_m(B_i + 1) \rceil + n)\). Denote \(B = (B_1, B_2, \ldots, B_k)\) and \(|\Psi| = (|\Psi_1|, |\Psi_2|, \ldots, |\Psi_k|)\). It then follows from (10) that the construction efficiency \(\rho_{s,k,m}(n, B, |\Psi|)\) for such a construction is given by

\[
\rho_{s,k,m}(n, B, |\Psi|) = \frac{\log_2 \sum_{i=1}^{k} |\Psi_i|}{m \sum_{i=1}^{k} ((n-1)\lceil \log_m(B_i + 1) \rceil + n)},
\]

where \(n, B,\) and \(|\Psi|\) satisfy the conditions in (A1)–(A3).

To maximize the construction efficiency in (27), it is clear that we should choose \(n\) as small as possible and choose each \(|\Psi_i|\) as large as possible. As such, we see from the conditions in (A1) and (A3) that we should choose \(n = \min\{2s+1, k\} + 1\) and \(|\Psi_i| = (m - 1)B_i + 1\) for \(i = 1, 2, \ldots, k\). It then follows from the condition in (A2) and (10) that the construction efficiency \(\rho_{s,k,m}(B)\) for such a construction is given by

\[
\rho_{s,k,m}(B) = \frac{\log_2 \sum_{i=1}^{k} (rB_j + 1)}{m \sum_{i=1}^{k} ((n-1)\lceil \log_m(B_i + 1) \rceil + n)},
\]

where \(r = m - 1, n = \min\{2s+1, k\} + 1,\) and \(B_1, B_2, \ldots, B_k\) satisfy the following condition in (B1):

(B1) \(B_1 = B_k = 1,\)

\[
1 \leq B_i \leq \begin{cases} 
\sum_{j=1}^{i-1} (rB_j + 1), & \text{if } 2 \leq i \leq s, \\
\sum_{j=-s}^{i-1} (rB_j + 1), & \text{if } s + 1 \leq i \leq k, 
\end{cases}
\]

and

\[
1 \leq B_i \leq \begin{cases} 
\sum_{j=1}^{i+s} (rB_j + 1), & \text{if } 1 \leq i \leq k - s, \\
\sum_{j=-s}^{i+s} (rB_j + 1), & \text{if } k - s + 1 \leq i \leq k - 1.
\end{cases}
\]

We denote \(\rho_{s,k,m}^*\) as the maximum value of \(\rho_{s,k,m}(B)\) over all choices of the buffer sizes \(B_1, B_2, \ldots, B_k\) that satisfy the condition in (B1). Intuitively, one would expect that the maximum construction efficiency \(\rho_{s,k,m}^*\) is achieved when each \(B_i\) is chosen as the maximum value under the constraint in (B1) (as such a choice of \(B_1, B_2, \ldots, B_k\) achieves the maximum buffer size \(\sum_{i=1}^{k} (rB_i + 1)\) under the constraint in (B1)), i.e., when \(B_1, B_2, \ldots, B_k\) are given by (B2) below:

(B2) If \(s + 1 \leq k \leq 2s\), then \(B_1 = B_k = 1\) and

\[
B_i = B_{i+1} - 1 = \sum_{j=1}^{i-1} (rB_j + 1) \text{ for } 2 \leq i \leq \lfloor k/2 \rfloor.
\]

On the other hand, if \(k \geq 2s + 1\), then \(B_1 = B_k = 1\) and

\[
B_i = B_{i+1} - 1 = \begin{cases} 
\sum_{j=1}^{i-1} (rB_j + 1), & \text{if } 2 \leq i \leq s, \\
\sum_{j=-s}^{i-1} (rB_j + 1), & \text{if } s + 1 \leq i \leq \lfloor k/2 \rfloor,
\end{cases}
\]

where \(r = m - 1\).

However, our numerical results show that this may not be the case as the construction complexity of the \(n\)-to-1 FIFO multiplexer is also increased in order to implement larger buffer sizes \(B_1, B_2, \ldots, B_k\), and the overall effect is a decrease in the construction efficiency when compared with the optimum choice of \(B_1, B_2, \ldots, B_k\). We denote \(\rho_{s,k,m} = \rho_{s,k,m}(B)\), where \(B_1, B_2, \ldots, B_k\) are given by (B2).

In Table I, we show our numerical results on the maximum construction efficiency \(\rho_{s,k,m}^*\), the construction efficiency \(\rho_{s,k,m}\), and the error rate \(E\) given by

\[
E = \frac{\rho_{s,k,m}^* - \rho_{s,k,m}}{\rho_{s,k,m}} \times 100\%.
\]

when we approximate \(\rho_{s,k,m}^*\) by \(\rho_{s,k,m}\). From Table I, we see that \(\rho_{s,k,m}\) is equal to \(\rho_{s,k,m}^*\) in some scenarios, especially when \(m\) is very small, and is close to \(\rho_{s,k,m}^*\) in many scenarios. The reason why \(\rho_{s,k,m}^*\) and \(\rho_{s,k,m}\) are not equal is due to the combinatorial nature and the discontinuity of the objective function (caused by the ceiling function in (28)) of such an optimization problem.

Furthermore, we see from Table I that both \(\rho_{s,k,m}^*\) and \(\rho_{s,k,m}\) decrease as \(s, k, m\) increases. In Section III-B, we will give closed-form expressions or bounds for \(\rho_{s,k,m}\) which support our observation that \(\rho_{s,k,m}\) decreases as \(s, k, m\) increases.

Finally, in Table I we also show the construction efficiency \(\hat{\rho}_k\) for the construction in [33] for \(k = 3, 5, 7, 9\). The construction efficiency \(\hat{\rho}_k\) for the construction in [33] can be obtained by using (27) with \(k = 2\ell - 1\) for some \(\ell \geq 2, m = 3, n = 4, B_1, B_2, \ldots, B_k\) given by (11), and \(|\Psi_1|, |\Psi_2|, \ldots, |\Psi_k|\) given by (12) as follows:

\[
\hat{\rho}_k = \frac{2 \log_2 (3 \cdot 2^{\ell-1} - 2)}{3(3\ell^2 + 13\ell + 4)}.
\]

We see that both \(\rho_{s,k,m}^*\) and \(\rho_{s,k,m}\) (with \(s = 1\) and \(m = 3\)) are larger than \(\hat{\rho}_k\) for \(k = 3, 5, 7, 9\), i.e., our construction is more effective than that in [33] in this special case.

B. The Construction Efficiency \(\tilde{\rho}_{s,k,m}\)

Recall that the construction efficiency \(\tilde{\rho}_{s,k,m}\) is given by

\[
\tilde{\rho}_{s,k,m} = \frac{\log_2 U_k}{M},
\]

where

\[
U_k = \sum_{i=1}^{k} (rB_i + 1),
\]

\[
M = m \sum_{i=1}^{k} [(n - 1)\lceil \log_m(B_i + 1) \rceil + n],
\]

with \(r = m - 1, n = \min\{2s+1, k\} + 1,\) and \(B_1, B_2, \ldots, B_k\) given by (B2).

To obtain closed-form expressions or bounds for \(\tilde{\rho}_{s,k,m}\), we first obtain closed-form expressions for the buffer sizes \(B_1, B_2, \ldots, B_k\) given by (B2) in the following theorem.
Theorem 8 Suppose that $1 \leq s \leq k - 1$, $m \geq 1$, and $r = m - 1$, and suppose that $B_1, B_2, \ldots, B_k$ are given by (B2).

(i) Suppose that $r = 0$. If $s + 1 \leq k \leq 2s$, then we have $B_1 = B_k = 1$ and

$$B_i = B_{k-i+1} = i - 1 \text{ for } 2 \leq i \leq [k/2].$$ 

(ii) Suppose that $s = 1$ and $r \geq 1$. Then we have

$$B_i = B_{k-i+1} = \begin{cases} i - 1, & \text{if } 2 \leq i \leq s, \\ s, & \text{if } s + 1 \leq i \leq [k/2]. \end{cases}$$ 

(iii) Suppose that $s \geq 2$ and $r \geq 1$. If $s + 1 \leq k \leq 2s$, then we have $B_1 = B_k = 1$ and

$$B_i = B_{k-i+1} = (r+1)i^{i-1} + \frac{1}{r}(r+1)i^{i-2} - 1 \text{ for } 2 \leq i \leq [k/2].$$ 

On the other hand, if $k \geq 2s + 1$, then we have $B_1 = B_k = 1$ and

$$B_i = B_{k-i+1} = \begin{cases} i - 1, & \text{if } 2 \leq i \leq s, \\ s, & \text{if } s + 1 \leq i \leq [k/2]. \end{cases}$$ 

where $\lambda_1, \lambda_2, \ldots, \lambda_s$ are the roots of the characteristic polynomial $p(z) = z^s - \sum_{j=0}^{s-1} r_j z^j$ associated with the $s$th-order nonhomogeneous linear difference equation with constant coefficients given by $B_i = \sum_{j=i-s}^{i-1} (r_j B_{j+1})$ for $s + 1 \leq i \leq [k/2]$, and $\alpha_1, \alpha_2, \ldots, \alpha_s$ can be obtained by solving the $s$ equations $B_1 = 1$ and $B_i = (r+1)i^{i-1} + \frac{1}{r}(r+1)i^{i-2} - 1$ for $i = 2, 3, \ldots, s$.

We need the following two lemmas for the proof of Theorem 8 whose proofs are given in Appendix B and Appendix C.
Lemma 9 Suppose that \( r \geq 1 \) and suppose that \( x_1 = 1 \) and \( x_i = \sum_{j=1}^{i-1} (rx_j + 1) \) for \( i \geq 2 \).
(i) \( x_i = (r + 1)x_{i-1} + 1 \) for \( i \geq 3 \).
(ii) \( x_i = (r + 1)^{i-1} + \frac{1}{r}((r + 1)^{i-2} - 1) \) for \( i \geq 2 \).

Lemma 10 Suppose that \( s \geq 2 \) and \( r \geq 1 \), and suppose that \( p \) is a polynomial given by \( p(z) = z^s - \sum_{j=0}^{s-1} r^j \).
(i) \( p \) has \( s \) complex roots and they are all distinct.
(ii) If \( s \) is odd, then \( p \) has one positive root and \( s-1 \) nonreal roots. On the other hand, if \( s \) is even, then \( p \) has one positive root, one negative root, and \( s-2 \) nonreal roots.
(iii) The positive root, say \( \lambda_+ \), of \( p \) lies in the open interval \( (r + 1 - 1/r^{s-1}, r + 1 - 1/(r + 1)^s) \), and \( \lambda_+ \approx r + 1 \) for sufficiently large \( s \) or \( r \). The other roots of \( p \) other than \( \lambda_+ \) lie in the annulus \( \{ z \in \mathbb{C} : \lambda_+/(\lambda_+ + 1) \leq |z| \leq \lambda_+ - 1 \} \).
Therefore, \( \lambda_+ \) is the root of \( p \) of the largest magnitude.

Proof. (Proof of Theorem 8) As we have from (B2) that \( B_i = B_{i-1} + 1 \) for \( 1 \leq i \leq \lfloor k/2 \rfloor \), it suffices to prove the results for \( B_i \) for \( 1 \leq i \leq \lfloor k/2 \rfloor \).
(i) By substituting \( r = 0 \) in (B2), we obtain (34) and (35).
(ii) First consider the case that \( k = 2 \). In this case, we have from (B2) that \( B_1 = 2 \) and hence (36) holds.
Now consider the case that \( k \geq 3 \). We will prove by induction on \( i \) that \( B_i = \sum_{j=0}^{i-1} r^j \) for \( 1 \leq i \leq \lfloor k/2 \rfloor \). Note that from (B2), we have \( B_1 = 1 \). Assume as the induction hypothesis that \( B_{i-1} = \sum_{j=0}^{i-2} r^j \) for some \( 2 \leq i \leq \lfloor k/2 \rfloor \). Then we have from (B2) (note that \( s = 1 \) in this case) and the induction hypothesis that
\[
B_i = rB_{i-1} + 1 = r \sum_{j=0}^{i-2} r^j + 1 = \sum_{j=0}^{i-1} r^j.
\]
(iii) First consider the case that \( s + 1 \leq k \leq 2s \). In this case, we have from (B2) that \( B_1 = 1 \) and \( B_i = \sum_{j=1}^{i-1} (rB_j + 1) \) for \( 2 \leq i \leq \lfloor k/2 \rfloor \), and hence (37) follows immediately from Lemma 9(ii).
Now consider the case that \( k \geq 2s + 1 \). Then we have from (B2) that \( B_1 = 1 \) and \( B_i = \sum_{j=1}^{i-1} (rB_j + 1) \) for \( 2 \leq i \leq s \), and hence it follows from Lemma 9(ii) that \( B_i = (r + 1)^i - 1 + \frac{1}{r}((r + 1)^{i-2} - 1) \) for \( 2 \leq i \leq s \).
From (B2), we also have the \( s \)-th order nonhomogeneous linear difference equation with constant coefficients given by \( B_i = \sum_{j=i-s}^{i-1} (rB_j + 1) \) for \( s + 1 \leq i \leq \lfloor k/2 \rfloor \). The characteristic polynomial \( p \) associated with this difference equation is given by \( p(z) = z^s - \sum_{j=0}^{s-1} z^j \). As we know from Lemma 10(i) that \( p \) has \( s \) roots, say \( \lambda_1, \lambda_2, \ldots, \lambda_s \), and they are all distinct, it then follows from well-established results in the theory of difference equations [60, Chapter 2] that \( B_i = \sum_{j=1}^{s} \alpha_j \lambda_j^i + \alpha_0 \) for \( 1 \leq i \leq \lfloor k/2 \rfloor \), where \( \alpha_j = \frac{-\alpha_{j+1}}{\lambda_j} \) is a particular solution to this difference equation, and \( \alpha_0, \alpha_1, \ldots, \alpha_s \) can be obtained by solving the \( s \) equations \( B_1 = 1 \) and \( B_i = (r + 1)^{i-1} + \frac{1}{r}((r + 1)^{i-2} - 1) \) for \( i = 2, 3, \ldots, s \).

In the following theorem (whose proof is given in Appendix D), we use the results in Theorem 8 to obtain closed-form expressions or bounds for \( \tilde{\rho}_{s,k,m} \).

Theorem 11 Suppose that \( 1 \leq s \leq k - 1, m \geq 1, \) and \( r = m - 1 \).
(i) Suppose that \( r = 0 \). If \( s + 1 \leq k \leq 2s \), then we have
\[
\tilde{\rho}_{s,k,m} = \frac{\log_2 k}{k(2k + 1)} \text{ and } U_k = (\sqrt{8M + 1} - 1)/4.
\]
On the other hand, if \( k \geq 2s + 1 \), then we have
\[
\tilde{\rho}_{s,k,m} = \frac{\log_2 k}{k(4s + 3)} \text{ and } U_k = M/(4s + 3).
\]
(ii) Suppose that \( s = 1 \) and \( k = 2 \). Then we have
\[
\tilde{\rho}_{s,k,m} = \frac{\log_2(r + 1) + 1}{10(r + 1)} \text{ and } U_k = M/5.
\]
(iii) Suppose that \( s = 1, k \geq 3, \) and \( r = 1 \). If \( k \) is odd, say \( k = 2\ell - 1 \) for some \( \ell \geq 2 \), then we have
\[
\tilde{\rho}_{s,k,m} = 2\log_2 \left( \frac{2\ell^2 + 3\ell}{4(3\ell^2 + 7\ell - 4\ell^2 + 12\ell + 16)} \right)
\]
and \( M/8 \leq U_k \leq (M/11)^2 \),
where \( \ell' \) is the unique integer such that \( 4\ell' + 1 \leq \ell \leq 4\ell' + 1 \).
(iv) Suppose that \( s = 1, k \geq 3, \) and \( r = 2 \). If \( k \) is odd, say \( k = 2\ell - 1 \) for some \( \ell \geq 2 \), then we have
\[
\tilde{\rho}_{s,k,m} = \frac{2\log_2 (3\cdot 2^{\ell + 1} - 2\ell - 7)}{3(3\ell^2 + 19\ell - 8)}
\]
and \( 2\sqrt{2M/9 + \log_3^2} \leq U_k \leq 2\sqrt{2M/9 + \log_3^3} \).
On the other hand, if \( k \) is even, say \( k = 2\ell \) for some \( \ell \geq 2 \), then we have
\[
\tilde{\rho}_{s,k,m} = \frac{2\log_2 (2\ell^2 - 2\ell - 8)}{3(3\ell^4 + 22\ell^3 + 44\ell + 16)} \text{, if } \ell \text{ is odd,}
\]
\[
\frac{2\log_2 (2\ell^2 - 2\ell - 8)}{3(3\ell^4 + 22\ell^3 + 44\ell + 16)} \text{, if } \ell \text{ is even,}
\]
and \( 2\sqrt{2M/9 + \log_5^2} \leq U_k \leq 2\sqrt{2M/9 + \log_5^5} \).
(v) Suppose that \( s = 1, k \geq 3, \) and \( r \geq 3 \). If \( k \) is odd, say \( k = 2\ell - 1 \) for some \( \ell \geq 2 \), then we have
\[
\frac{\log_2 U_k}{M_1} \leq \tilde{\rho}_{s,k,m} \leq \frac{\log_2 U_k}{M_2},
\]
where
\[
U_k = ((r + 1)r^{\ell + 1} - 2r^2 - (2\ell - 1)(r - 1))/((r - 1)^2),
\]
\[
M_1 = 3(\ell + 3)^2(r + 1)\log_4 r,
\]
\[
M_2 = 3(\ell - 1)^2(r + 1)\log_4 r,
\]
and we have
\[
2\sqrt{2M_2 \log_2 r/(3(r + 1))} - 3\log_2 r \leq U_k \leq 2\sqrt{2M_2 \log_2 r/(3(r + 1)) + \log_2(3r)}.
\]
On the other hand, if \( k \) is even, say \( k = 2\ell \) for some \( \ell \geq 2 \), then (50) holds with

\[
\begin{align*}
U_k &= (2\ell + 2 - 2\ell^2 - 2\ell(r - 1)) / (r - 1)^2, \\
M_1 &= 3(\ell + 4)^2(r + 1) \log_4 r, \\
M_2 &= 3(\ell - 1)^2(r + 1) \log_4 r,
\end{align*}
\]

and we have

\[
2\sqrt{2M \log_2 \rho(r/(3(r + 1))) - 4 \log_2 r + 1} \leq U_k \leq 2\sqrt{2M \log_2 \rho(r/(3(r + 1))) + \log_2(9r/2)},
\]

(vi) Suppose that \( s \geq 2, r \geq 1, \) and \( s + 1 \leq k \leq 2s \). If \( k \) is odd, say \( k = 2\ell - 1 \) for some \( \lceil s/2 \rceil + 1 \leq \ell \leq s \), then (50) holds with

\[
\begin{align*}
U_k &= (r + 1)^\ell + (r + 2 + 2/r)(r + 1)^{\ell - 2} - 2/r, \\
M_1 &= k(\ell + 4/\log_{k+1}(r + 1))^{2}(r + 1) \log_{k+1}(r + 1), \\
M_2 &= k(\ell - 1)^2(r + 1) \log_{k+1}(r + 1),
\end{align*}
\]

and we have

\[
2\sqrt{M \log_2(k+1) \log_2(r/(k(r + 1))) - 4 \log_2(k+1)} \leq U_k \leq 2\sqrt{M \log_2(k+1) \log_2(r/(k(r + 1))) + \log_2(3(r + 1))}.
\]

On the other hand, if \( k \) is even, say \( k = 2\ell \) for some \( \lceil (s + 1)/2 \rceil \leq \ell \leq s \), then (50) holds with

\[
\begin{align*}
U_k &= 2(r + 1)^\ell + 2(r + 1)^{\ell - 1} - 2/r, \\
M_1 &= k(\ell + 4/\log_{k+1}(r + 1))^{2}(r + 1) \log_{k+1}(r + 1), \\
M_2 &= k(\ell - 1)^2(r + 1) \log_{k+1}(r + 1),
\end{align*}
\]

and we have

\[
2\sqrt{M \log_2(k+1) \log_2(r/(k(r + 1))) - 4 \log_2(k+1)} + 1 \leq U_k \leq 2\sqrt{M \log_2(k+1) \log_2(r/(k(r + 1))) + \log_2(3(r + 1))}.
\]

Remark 12 (ii) It is clear that the closed-form expressions or bounds for \( \bar{\rho}_{s,k,m} \) in Theorem 11 decrease as \( s, k, m \) increases, and this supports our observation in Section III-A that \( \bar{\rho}_{s,k,m} \) decreases as \( s, k, m \) increases.

(ii) For the special case that \( s = 1, k = 2\ell - 1 \) for some \( \ell \geq 2, \) and \( m = 3, \) it can be seen from (46) and (30) that \( \bar{\rho}_{s,k,m} > \tilde{\rho}_k, \) and this confirms our observation in Section III-A that our construction is more effective than that in [33] in this special case.

(iii) In Theorem 11, we have also obtained closed-form expressions or bounds for the buffer size \( U_k \) in terms of \( M, \) and we can see that \( U_k = \Theta(M^\ell) \) for some \( 1 \leq c \leq 2 \) or \( U_k = 2^{\Theta(\sqrt{M})}. \) This shows that there is still a gap between the buffer size achieved in our construction and the theoretical upper bound \( 2^M. \)

(iv) We note that the case that \( s \geq 2, r \geq 1, \) and \( k \geq 2s+1 \) is not considered in Theorem 11. This is because in the closed-form expression for \( B_1, B_2, \ldots, B_k \) in (38), \( \lambda_1, \lambda_2, \ldots, \lambda_s \) and \( \alpha_1, \alpha_2, \ldots, \alpha_s \) cannot be easily expressed in terms of \( s, k, \) and \( m. \) However, for this case we know from Lemma 10(iii) that the positive root \( \lambda_+ \) of the characteristic polynomial \( p \) satisfies \( \lambda_+ > r + 1 - 1/r^{s-1} \geq 1 \) and any root \( \lambda \) of \( p \) other than \( \lambda_+ \) satisfies \( |\lambda| \leq \lambda_+ - r < 1 - r/(r + 1)^s < 1, \) and hence \( \lambda_+ \) is very large and \( |\lambda_+| \) is very small when \( s \) is very large.

For sufficiently large \( s \) or \( r, \) we know from Lemma 10(iii) that \( \lambda_+ \approx r + 1, \) and if we approximate \( B_i \) by \( (r + 1)^i \) for \( 1 \leq i \leq \lceil k/2 \rceil, \) then we can obtain rough estimates for \( \bar{\rho}_{s,k,m} \) and \( U_k \) as follows (see Appendix E for details):

\[
\begin{align*}
\rho_{s,k,m} &\approx \log_2(2s + 2)/((2s + 1)(r + 1)), \\
U_k &\approx 2\sqrt{M \log_2(2s + 2)/((2s + 1)(r + 1)),}
\end{align*}
\]

These rough estimates are pretty good in the order sense for the case that \( s = 1, k \geq 3, \) and \( r \geq 2 \) as (67) and (68) reduce to the results in Theorem 11(iv) (in the order sense) when \( s = 1, k \geq 3, \) and \( r = 2, \) and reduce to the results in Theorem 11(v) (in the order sense) when \( s = 1, k \geq 3, \) and \( r \geq 3. \)

IV. Conclusion

In this paper, we first showed that if the conditions in (A1)–(A3) are satisfied, then the feedback system in Figure 1 can be operated as an optical priority queue with buffer size \( U_k \) under the priority-based routing policy in (R1)–(R3). Then we introduced the construction efficiency as a performance measure for SDL constructions of optical priority queues, and showed that the inequalities in (A1) and (A3) should hold with equality and the buffer sizes \( B_1, B_2, \ldots, B_k \) should satisfy the condition in (B1) in order to maximize the construction efficiency for our constructions. Finally, we obtained closed-form expressions or bounds for the construction efficiency for the special case that \( B_1, B_2, \ldots, B_k \) are given by (B2). Our results show that our construction is more effective than the best construction in [33] in the literature, and can achieve a buffer size of \( 2^{\Theta(\sqrt{M})}. \) by using a feedback system consisting of an \( (M + 2) \times (M + 2) \) (bufferless) optical crossbar switch and \( M \) fiber delay lines.

APPENDIX A

The Feedforward System in Figure 4 Can Be Replaced by the Feedback System in Figure 5

The self-routing scheme for the construction of an optical \( n \)-to-1 FIFO multiplexer with buffer size \( n^\ell - 1 \) in [12] by using the feedforward system in Figure 4 can be described as follows. Consider a packet, say packet \( p, \) that arrives at the FIFO multiplexer at time \( t. \) Suppose that the virtual delay of packet \( p \) at time \( t \) is \( x, \) i.e., packet \( p \) sees \( x \) packets in the FIFO multiplexer upon its arrival at time \( t. \) If \( x = n^{\ell - 1} + i \) for some \( 1 \leq i \leq n - 1, \) then we route packet \( p \) to loss link \( i \) at time \( t \) by using the \( n \times (2n - 1) \) crossbar switch. On the other hand, if \( 0 \leq x \leq n^{\ell - 1} - 1, \) then we write the \( n \)-ary representation of \( x \) as \( x = \sum_{i=0}^{n-1} I_{i}(x)n^{i-1}, \) where \( 0 \leq \ell \leq \ell, \) \( 1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq \ell, \) and \( 1 \leq I_{i}(x) \leq n - 1 \) for \( j = 1, 2, \ldots, \ell. \) We route packet \( p \) to the fiber with delay \( I_{i}(x)n^{i-1} \) at time \( t \) by using the \( n \times (2n - 1) \) crossbar switch and the first \( i_1 - 1 \) \( n \times n \) crossbar switches, to the fiber with delay \( I_{i}(x)n^{i-1} \) at time \( t + \sum_{j=1}^{i-1} I_{i}(x)n^{j-1} \) by using the next \( i_2 - i_1 \) \( n \times n \) crossbar switches, and finally to the
departure link at time $t + ∑_{j=1}^{ℓ} I_j(x) n_j - 1 = t + x$ by using the last $ℓ - i$, $n \times n$ crossbar switches and the $n \times 1$ crossbar switch. Therefore, packet $p$ is routed to the right place at the right time.

As it is clear that by using the feedback system in Figure 5, packet $p$ can also be self-routed to the loss link or the departure link through fibers with the same delays at the times as described above, it follows that the feedforward system in Figure 4 can be replaced by the feedback system in Figure 5.

**Appendix B**

**Proof of Lemma 9**

(i) Suppose that $i \geq 3$. Then we have

$$x_i = \sum_{j=1}^{i-1} (rx_j + 1) = \sum_{j=1}^{i-2} (rx_j + 1) + rx_{i-1} + 1$$

$$= x_{i-1} + rx_{i-1} + 1 = (r + 1)x_{i-1} + 1.$$  

(ii) We prove (ii) by induction on $i$. Clearly, we have $x_2 = rx_1 + 1 = r + 1 = r + 1$. Assume as the induction hypothesis that $x_{i-1} = (r + 1)^i - 1$ for some $i \geq 3$. Then we have from (i) and the induction hypothesis that

$$x_i = (r + 1)x_{i-1} + 1 = (r + 1)\left[(r + 1)^{i-2} + \frac{1}{r}(r + 1)^{i-3} - 1\right] + 1$$

$$= (r + 1)^{i-1} + \frac{1}{r}(r + 1)^{i-2} - 1.$$ 

**Appendix C**

**Proof of Lemma 10**

(i) As $p$ is clearly a polynomial of degree $s$, it follows from the fundamental theorem of algebra [52, Theorem 16.22] that $p$ has $s$ complex roots.

We show that the roots of $p$ are distinct by contradiction. Assume on the contrary that $p$ has a repeated root, say $λ$. Let $f$ be a polynomial given by $f(z) = (z - 1)p(z)$. Then we have

$$f(z) = (z - 1)p(z) = (z - 1)\left(z^s - ∑_{j=0}^{s-1} rz^j\right)$$

$$= z^{s+1} - (r + 1)z^s + r. \quad (69)$$

As it is clear that $λ$ is also a repeated root of $f$, we must have $f'(λ) = 0$. Thus, we see from (69) that $(s + 1)λ^s - (r + 1)sλ^{s-1} = 0$, i.e., $λ = 0$ or $λ = (r+1)s$. Since $0$ cannot be a root of $p$ (as $p(0) = −r ≠ 0$), we must have

$$λ = \frac{(r+1)s}{s}. \quad (70)$$

From $λ = \frac{(r+1)s}{s + 1} ≠ 0$ (as $rs ≠ 1$), we have

$$p(λ) = λ^s - ∑_{j=0}^{s-1} rλ^j = λ^s - \frac{r(λ^s - 1)}{λ - 1}$$

$$= λ^s - \frac{r(s+1)(λ^{s-1} - 1)}{rs - 1} = \frac{r(s+1) - (r+1)λ^s}{rs - 1},$$

and it then follows from $p(λ) = 0$ that

$$λ^s = \frac{r(s+1)}{r + 1}. \quad (71)$$

If $r = 1$, then we see from $λ = \frac{(r+1)s}{s + 1}$ in (70) and $s ≥ 2$ that

$$λ^s = \frac{r(s+1)}{r + 1}$$

$$= \left(\frac{2s}{s + 1}\right) - \frac{s + 1}{2} = \left(1 + \frac{s - 1}{s + 1}\right) - \frac{s + 1}{2} = \frac{(s - 1)^2}{2(s + 1)} > 0,$$

and we have reached a contradiction to (71) in this case. On the other hand, if $r ≥ 2$, then we see from $λ = \frac{(r+1)s}{s + 1}$ in (70) and $s ≥ 2$ that

$$λ^s = \left(\frac{(r+1)s}{s + 1}\right) - \frac{(2+1)2^s}{2(s + 1)} = 2^s ≥ s + 1 > \frac{r(s+1)}{r + 1},$$

and we have also reached a contradiction to (71) in this case.

(ii) To prove (ii), we need Descartes’ rule of signs [53]–[56], which says that the number $z_+(f)$ of positive roots (counting multiplicities) of a nonzero polynomial $f$ with real coefficients is at most equal to the number $v(f)$ of changes of signs in the sequence of the polynomial’s coefficients (omitting the zero coefficients), and that the difference between these two numbers is even, i.e.,

$$v(f) - z_+(f) \text{ is a nonnegative even integer.} \quad (72)$$

Note that it is easy to see from (72) and $z_+(f) ≥ 0$ that

$$v(f) ≤ 1, \text{ then } z_+(f) = v(f). \quad (73)$$

As it is clear that $v(p) = 1$, we have from (73) that $z_+(p) = v(p) = 1$, i.e., $p$ has exactly one positive root. Furthermore, $0$ cannot be a root of $p$ (as $p(0) = −r ≠ 0$).

Let $f(z) = (z - 1)p(z)$ and let $g(z) = f(-z) = -(z + 1)p(-z)$. Then it is easy to see that a positive number $λ$ is a root of $g$ if and only if $-λ$ is a root of $p$.

First consider the case that $s$ is odd. In this case, we see from (69) that $g(z) = z^{s+1} + (r + 1)z^s + r$. As it is clear that $v(g) = 0$, we have from (73) that $z_+(g) = v(g) = 0$, i.e., $g$ has no positive roots, or, equivalently, $p$ has no negative roots. Therefore, we conclude from (i) and the results above that $p$ has one positive root and $s - 1$ nonreal roots in this case.

Now consider the case that $s$ is even. In this case, we see from (69) that $g(z) = z^{s+1} - (r + 1)z^s + r$. As it is clear that $v(g) = 1$, we have from (73) that $z_+(g) = v(g) = 1$, i.e., $g$ has exactly one positive root, or, equivalently, $p$ has exactly one negative root. Therefore, we conclude from (i) and the results above that $p$ has one positive root, one negative root, and $s - 2$ nonreal roots in this case.

(iii) As $p(r) = −∑_{j=0}^{s-2} r^j + 1 < 0$ (note that $s ≥ 2$) and $p(r + 1) = 1 > 0$, we have from the intermediate-value theorem for continuous functions [52, Theorem 4.33] that $r < λ_+ < r + 1$. From (69) and $p(λ_+) = 0$, we see that

$$λ_+^{s+1} - (r + 1)λ_+^s + r = (λ_+ - 1)p(λ_+) = 0,$$
and it then follows from $\lambda_+ \neq 0$ that
\[
\lambda_+ = r + 1 - \frac{r}{\lambda_+^s}.
\] (74)

Since $s \geq 2$, $r \geq 1$, and we have proved that $r < \lambda_+ < r+1$, we deduce from (74) that $r + 1 - 1/r^{s-1} < \lambda_+ < r + 1 - r/(r+1)^s$ and $\lambda_+ \approx r + 1$ for sufficiently large $s$ or $r$.

To show that the other roots of $p$ other than $\lambda_+$ lie in the annulus $\{z \in C : \lambda_+/(\lambda_+ + 1) \leq |z| \leq \lambda_+ - r\}$, we need Esterlöm-Kakeya theorem [57]–[59], Theorem 4.4, which says that if $f(z) = \sum_{i=0}^{\infty} b_i z_i$ is a polynomial of degree $\ell$ with positive coefficients, i.e., $b_i > 0$ for $i = 0, 1, \ldots, \ell$, then all the roots of $f$ lie in the annulus $\{z \in C : \min_{0 \leq i \leq \ell-1} \frac{b_i}{b_{i+1}} \leq |z| \leq \max_{0 \leq i \leq \ell-1} \frac{b_i}{b_{i+1}}\}$. It is easy to see that $p(z)$ can be written as $p(z) = (z - \lambda_+)q(z)$, where $q(z) = \sum_{i=0}^{\infty} b_i z_i$ with $b_{s-1} = 1$, $b_i = \lambda_i b_{i-1} - r$ for $i = s-2, s-3, \ldots, 0$, and $\lambda_i b_0 - r = 0$. In the following, we prove by induction on $i$ that
\[
b_i = \frac{r}{\lambda_+^{i+1}} \text{ for } i = 0, 1, \ldots, s-1.
\] (75)

Note that from $\lambda_i b_0 - r = 0$, we have $b_0 = r/\lambda_+$. Assume as the induction hypothesis that (75) holds for some $0 \leq i \leq s-2$. Then we have from $b_i = \lambda_i b_{i+1} - r$ and the induction hypothesis that
\[
b_{i+1} = \frac{1}{\lambda_+} (b_i + r) = \frac{1}{\lambda_+} \sum_{j=0}^{i} \frac{r}{\lambda_+^{j+1}} + \frac{r}{\lambda_+} + \sum_{j=0}^{i+1} \frac{r}{\lambda_+^{j+1}}.
\]

From $b_i = \lambda_i b_{i+1} - r$ for $0 \leq i \leq s-2$, $0 < b_0 < b_1 < \cdots < b_{s-1}$ (by (75)), $b_i = \frac{r}{\lambda_+^{i+1}}(\lambda_+ + 1)$ in (75), and $b_{s-1} = 1$, we obtain
\[
\min_{0 \leq i \leq s-2} \frac{b_i}{b_{i+1}} = \min_{0 \leq i \leq s-2} \left( \frac{\lambda_+ - r}{\lambda_+} \right) = \lambda_+ - r = \frac{\lambda_+}{\lambda_+ + 1},
\] (76)

and
\[
\max_{0 \leq i \leq s-2} \frac{b_i}{b_{i+1}} = \max_{0 \leq i \leq s-2} \left( \frac{\lambda_+ - r}{\lambda_+} \right) = \lambda_+ - r = \lambda_+ - r.
\] (77)

Therefore, we have from Esterlöm-Kakeya theorem, (76), and (77) that all the roots of $q$, i.e., all the roots of $p$ other than $\lambda_+$, lie in the annulus $\{z \in C : \lambda_+/(\lambda_+ + 1) \leq |z| \leq \lambda_+ - r\}$.

**Appendix D**

**Proof of Theorem 11**

(i) Note that $m = r + 1 = 1$ and we have from (32) that $U_k = \sum_{i=1}^{k} B_i = k$. First consider the case that $s+1 \leq k \leq 2s$. In this case, we have $n = k+1$. As it is easy to see from (34) that $1 < B_i + 1 < k + 1$ and hence $\log^2(B_i + 1) = 1$ for $1 \leq i \leq k$, we have from (33) that $M = \sum_{i=1}^{k} (k \cdot 1 + k + 1) = k(2k+1)$. Thus, (39) holds.

Now consider the case that $k \geq 2s+1$. In this case, we have $n = 2s + 2$. As it is easy to see from (35) that $1 < B_i + 1 < 2s + 2$ and hence $\log_{2s+2}(B_i + 1) = 1$ for $1 \leq i \leq k$, we have from (33) that $M = \sum_{i=1}^{k} ((2s+1) \cdot 1 + (2s + 2)) = k(4s + 3)$. Thus, (40) holds.

(ii) Note that $n = k + 1 = 3$ and we have from (B2) that $B_1 = B_2 = 1$. It follows from (32) that $U_k = \sum_{i=1}^{k} (r + 1) = 2(r+1)$ and from (33) that $M = (r+1) \sum_{i=1}^{k} (2(\log_3 2) + 3) = 10(r+1)$. Thus, (41) holds.

(iii) Note that $m = r + 1 = 2, n = s + 2 = 4$, and we have from (36) that $B_i = B_{k-i+1} = 1$ for $1 \leq i \leq [k/2]$. First consider the case that $k$ is odd, say $k = 2\ell - 1$ for some $\ell \geq 2$. In this case, we have $U_k = 2 \sum_{i=1}^{\ell} (i+1) + (\ell+1) = \ell^2 + 2\ell - 1$ and
\[
M = 2 \left[ 2 \sum_{i=1}^{\ell-1} (3\log_4(i+1) + 1) + (3\log_4(i+1) + 4) \right] = 12 \left[ \sum_{j=1}^{\ell'} j(4^j - 4^{j-1}) + (\ell' + 1)(\ell - 4^{\ell'}) \right] + 6(\log_4(i+1) + 8(2\ell - 1)
\]
\[
= 4(3\ell + 7)\ell - 4^{\ell+1} + 6(\log_4(i+1) - 4),
\] (78)

where $\ell'$ is the unique integer such that $4^\ell + 1 \leq \ell \leq 4^{\ell+1}$. Thus, (42) holds. From (78) and $\ell' \leq \log_4(i+1) \leq (\ell - 2)/2$ (note that $\ell \geq 2$), we obtain
\[
M \leq 4(3\ell + 7)\ell - 4\ell + 6(\ell + 2) - 4
\]
\[
\leq 6(\ell - 2) + 24\ell + 3(\ell + 2) - 4
\]
\[
= 6\ell^2 + 15\ell + 2 \leq 8U_k
\] (79)
\[
M \geq 4(3\ell + 7)\ell - 16(\ell - 1) + 6(\ell' + 1) - 4
\]
\[
\geq 12(\ell + 1) \geq 12\sqrt{U_k}.
\] (80)

Thus, (43) follows from (79) and (80).

Now consider the case that $k$ is even, say $k = 2\ell$ for some $\ell \geq 2$. In this case, we have $U_k = 2 \sum_{i=1}^{\ell} (i+1) = \ell^2 + 3\ell$ and
\[
M = 2 \cdot \sum_{i=1}^{\ell} (3\log_4(i+1) + 1)
\]
\[
= 12 \left[ \sum_{j=1}^{\ell'} j(4^j - 4^{j-1}) + (\ell' + 1)(\ell + 1 - 4^{\ell'}) \right] + 16\ell
\]
\[
= 4(3\ell + 7)\ell - 4^{\ell+1} + 12\ell + 16,
\] (81)

where $\ell'$ is the unique integer such that $4^\ell \leq \ell \leq 4^{\ell+1} - 1$. Thus, (44) holds. From (81) and $\ell' \leq \log_4(i+1) \leq (\ell - 1)/2$, we obtain
\[
M \leq 4(3\ell + 7)\ell - 4(\ell + 1) + 12\ell + 16
\]
\[
\leq (6(\ell - 1) + 24\ell + 6(\ell + 1) + 12
\]
\[
= 6\ell^2 + 24\ell + 6 \leq 8U_k
\] (82)
\[
M \geq 4(3\ell + 7)\ell - 16(\ell - 1) + 12\ell + 16
\]
\[
\geq 12\ell + 16 \geq 12\sqrt{U_k}.
\] (83)

Thus, (45) follows from (82) and (83).

(iv) Note that $m = r + 1 = 3, n = 2s + 2 = 4$, and we have from (36) that $B_i = B_{2s+2-i+1} = 1$ for $1 \leq i \leq [k/2]$. First consider the case that $k$ is odd, say $k = 2\ell - 1$ for some $\ell \geq 2$. In this case, we have $U_k = 2 \sum_{i=1}^{\ell} (2^{i+1} - 1) + (2^{i+1} - 1) =


Thus, (46) holds. As it is easy to see from \( \ell \geq 2 \) that

\[
3 \cdot 2^\ell - 2 \ell - 7 \quad \text{and we have from } \log_4(B_i + 1) = i/2 \text{ for } 1 \leq i \leq \lfloor k/2 \rfloor \quad \text{that}
\]

\[
M = 3 \left[ \sum_{i=1}^{\ell-1} (3i/2 + 4) + (3[\ell/2] + 4) \right] \\
= 3(3\ell^2 + 19\ell - 8)/2.
\]

Thus, (46) holds. As it is easy to see from \( \ell \geq 2 \) that

\[
3 \cdot 2^\ell \leq U_k \leq 3 \cdot 2^{\ell+1} \tag{84}
\]

\[
9(\ell + 1)^2/2 \leq M \leq 9(\ell + 4)^2/2, \tag{85}
\]

it then follows from (84) and \( \sqrt{2M}/9 - 4 \leq \ell \leq \sqrt{2M}/9 - 1 \) in (85) that (47) holds.

Now consider the case that \( k \) is even, say \( k = 2\ell \) for some \( \ell \geq 2 \). In this case, we have \( U_k = 3 \ell \sum_{i=1}^{\ell-1} (2^{i-1} - 1) = 2^{2\ell} - 2\ell - 8 \) and

\[
M = 3 \cdot 2 \sum_{i=1}^{\ell} (3[\log_4 2^i] + 4) = 18 \sum_{i=1}^{\ell} [i/2] + 24\ell
\]

\[
= \begin{cases} 3(3\ell^2 + 22\ell + 3)/2, & \text{if } \ell \text{ is odd}, \\
3\ell(3\ell + 22)/2, & \text{if } \ell \text{ is even} \end{cases}
\]

Thus, (48) holds. As it is easy to see from \( \ell \geq 2 \) that

\[
5 \cdot 2^\ell \leq U_k \leq 2^{\ell+3} \tag{86}
\]

\[
9(\ell + 2)^2/2 \leq M \leq 9(\ell + 4)^2/2, \tag{87}
\]

it then follows from (86) and \( \sqrt{2M}/9 - 4 \leq \ell \leq \sqrt{2M}/9 - 2 \) in (87) that (49) holds.

(v) Note that \( n = 2s + 2 = 4 \) and we have from (36) that \( B_i = B_{k-i+1} = (r^i - 1)/(r - 1) \) for \( 1 \leq i \leq \lfloor k/2 \rfloor \). First consider the case that \( k \) is odd, say \( k = 2\ell - 1 \) for some \( \ell \geq 2 \).

A straightforward calculation shows that (51) holds.

For \( 1 \leq i \leq \lfloor k/2 \rfloor \), it is easy to see that \( r^{i-1} - 1 \leq B_i \leq r^{\ell - 1} \), and it follows that \( \log_4(B_i + 1) \leq \log_4(B_i + 1) \leq i \log_4 r + 1 \), and \( \log_4(B_i + 1) \geq \log_4(B_i + 1) \geq (i - 1) \log_4 r \). As such, we have

\[
M = \sum_{i=1}^{\ell} 3(i \log_4 r + 1) + 4(3\ell \log_4 r + 1) + 4) \\
= 3\ell^3 \log_4 r + 7(2\ell - 1) \leq 3(\ell + 3)^2 \log_4 r, \tag{88}
\]

where we have used \( \log_4 r \geq \log_4 3 \geq 0.79 \), and we also have

\[
M \geq 3(\ell - 1)^2 \log_4 r + 4(2\ell - 1) \geq 3(\ell - 1)^2 \log_4 r. \tag{89}
\]

Thus, (50) follows from (31), (88), (89), (52), and (53). As it can be seen from (51), \( \ell \geq 2 \), and \( r > 3 \) that

\[
U_k \geq [(r^2 + r - 2)r^\ell - (2\ell - 1)(r - 1)]/(r - 1)^2 \\
= r^\ell + (3r^\ell - 2\ell + 1)/(r - 1) \geq r^\ell \tag{90}
\]

\[
U_k \leq (r + 1)r^\ell+1/(r - 1)^2 \leq 3r^\ell \tag{91}
\]

it then follows from (90), (91), \( \ell \geq \sqrt{2M}/(3(r + 1) \log_2 r) - 3 \) in (88), and \( \ell \leq \sqrt{2M}/(3(r + 1) \log_2 r) + 1 \) in (89) that (54) holds.

Now consider the case that \( k \) is even, say \( k = 2\ell \) for some \( \ell \geq 2 \). A straightforward calculation shows that (55) holds. Furthermore, we have

\[
M \geq 3(\ell - 1)^2 \log_4 r, \tag{92}
\]

where we have used \( \log_4 r \geq \log_4 3 \geq 0.79 \), and we also have

\[
M \geq 3(\ell - 1)^2 \log_4 r. \tag{93}
\]

Thus, (50) follows from (31), (92), (93), (56), and (57). As it can be seen from (55), \( \ell \geq 2 \), and \( r \geq 3 \) that

\[
U_k \geq (2r^2 - 2r + (4\ell - 2)/r - 1) \geq 2r^\ell \tag{94}
\]

\[
U_k \leq 2r^\ell + (r - 1)^2 \leq 9r^\ell \tag{95}
\]

it then follows from (94), (95), \( \ell \geq \sqrt{2M}/(3(r + 1) \log_2 r) - 4 \) in (92), and \( \ell \leq \sqrt{2M}/(3(r + 1) \log_2 r) + 1 \) in (93) that (58) holds.

(vi) Note that \( n = k + 1 \) and we have from (37) that \( B_1 = B_{k+1} = (r + 1)^{k+1} + 1 = (r + 1)2^{(k+1)/2} - 1 \) for \( 2 \leq i \leq \lfloor k/2 \rfloor \). First consider the case that \( k \) is odd, say \( k = 2\ell - 1 \) for some \( \ell \geq 2 \). A straightforward calculation shows that (59) holds.

For \( 1 \leq i \leq \lfloor k/2 \rfloor \), it is easy to see that \( (r + 1)^{i-1} - 1 \leq B_i \leq (k + 1)(r + 1)^{i-1} - 1 \), and it follows that \( \log_{k+1}(B_i + 1) < \log_{k+1}(B_{i+1} + 1) + 1 \leq (i - 1) \log_{k+1}(r + 1) + 2 \) and \( \log_{k+1}(B_{i+1} + 1) \geq \log_{k+1}(B_{i+1} + 1) + (i - 1) \log_{k+1}(r + 1) \).

As such, we have

\[
M \geq \sum_{i=1}^{\ell} \log_{k+1}(B_i + 1)/(r + 1) + 2 + k + 1 \\
= k(\ell - 1)^2 \log_{k+1}(r - 1) + (3k + 1)(2\ell - 1) \\
\leq k(\ell - 1)^2 \log_{k+1}(r + 1)^2 \log_{k+1}(r + 1), \tag{96}
\]

and we also have

\[
M \geq \sum_{i=1}^{\ell} \log_{k+1}(B_i + 1)/(r + 1) + 2 + k + 1 \\
= k(\ell - 1)^2 \log_{k+1}(r + 1) + (k + 1)(2\ell - 1) \\
\geq k(\ell - 1)^2 \log_{k+1}(r + 1). \tag{97}
\]

Thus, (50) follows from (31), (96), (97), (60), and (61). From (59), \( \ell \geq \lfloor s/2 \rfloor + 1 \geq 2 \), and \( r \geq 1 \), we can see that

\[
(r + 1)^{\ell} \leq U_k \leq 3(r + 1)^{\ell}. \tag{98}
\]

As we have \( \ell \geq \sqrt{M \log_2 (k + 1)/(k + 1) \log_2 (r + 1) - 4 \log_2 (k + 1)/(k + 1) \log_2 (r + 1) + 1} \) in (96) and we also have \( \ell \leq \sqrt{M \log_2 (k + 1)/(k + 1) \log_2 (r + 1) + 1} \) in (97), it then follows from (98) that (62) holds.
Now consider the case that $k$ is even, say $k = 2\ell$ for some $\frac{(s + 1)/2}{\ell} \leq s \leq s$. A straightforward calculation shows that (63) holds. Furthermore, we have

\[
\frac{M}{r+1} \leq 2 \sum_{i=1}^{\ell} [k(i - 1) \log_{k+1}(r+1) + 2] + k + 1
\]

\[
= k\ell(\ell - 1) \log_{k+1}(r+1) + (3k + 1) \cdot 2\ell
\]

\[
\leq k\ell(\ell + 4/\log_{k+1}(r+1))^2 \log_{k+1}(r+1),
\]  

(99)

and we also have

\[
\frac{M}{r+1} \geq 2 \sum_{i=1}^{\ell} (k(i - 1) \log_{k+1}(r+1) + k + 1)
\]

\[
= k\ell(\ell - 1) \log_{k+1}(r+1) + (k + 1) \cdot 2\ell
\]

\[
\geq k(\ell - 1)^2 \log_{k+1}(r+1).
\]  

(100)

Thus, (50) follows from (31), (99), (100), (64), and (65). From (63), $\ell \geq \frac{(s + 1)/2}{\ell} \geq 2$, and $r \geq 1$, we can see that

\[
2(r+1)^{1/2} \leq U_k \leq 3(r+1)^{1/2}.
\]  

(101)

As we have $\ell \geq \sqrt{M \log_{k}(k+1)/[(k+1) \log_{k}(r+1) - 4 \log_{k}(k+1)/\log_{k}(r+1)]}$ in (99) and we also have $\ell \leq \sqrt{M \log_{k}(k+1)/[(k+1) \log_{k}(r+1) + 1]}$ in (100), it then follows from (101) that (66) holds.

**APPENDIX E**

**PROOF OF (67) AND (68)**

Suppose that $B_i = B_{k+i-1} \approx (r+1)^i$ for $1 \leq i \leq [k/2]$. We only prove (67) and (68) for the case that $k$ is odd as the proof for the case that $k$ is even is similar. Suppose that $k$ is odd, say $k = 2\ell - 1$ for some $\ell \geq 2$. Note that $n = 2s + 2$ (as $k \geq 2s + 1$). Then we have from (32) that

\[
U_k \approx 2 \sum_{i=1}^{\ell-1} (r(r+1)^i + 1) + r(r+1)^{i+1}
\]

\[
= 2(r+1)^{i+1} - (r+1) + r(r+1)^{i+1} + 2\ell - 1
\]

\[
\approx (r+1)^{i+1} + 2\ell - 1,
\]  

(102)

and we have from (33) that

\[
\frac{M}{r+1} \approx 2 \sum_{i=1}^{\ell-1} [(2s + 1) \log_{2s+2}(r+1)^i + 2s + 2]
\]

\[
+ [(2s + 1) \log_{2s+2}(r+1)^i + 2s + 2]
\]

\[
= (2s + 1)^{\ell} \log_{2s+2}(r+1)^i + (2s + 2)(2\ell - 1)
\]

\[
\approx (2s + 1)^{\ell} \log_{2s+2}(r+1)^i.
\]  

(103)

Thus, (67) follows from (32), (102), and (103). From (102) and $\ell \leq \sqrt{M \log_{k}(2s + 2)/((2s + 1) \log_{k}(r+1)^i)}$ in (103), we see that (68) holds.

**REFERENCES**


