COM 5110
ccRandom Processes for Communications - Spring 2018
Homework 3
(Due date: June 11, 2018)

1. (10 points) Consider the system function $H(f)$ of an brick-wall low-pass filter defined by

$$H(f) = \begin{cases} 1, & \text{if } |f| \leq W, \\ 0, & \text{if } |f| > W. \end{cases}$$

Find the impulse response function $h(t)$. Also derived the equation:

$$\int_0^\infty \text{sinc}(x)\, dx = \frac{1}{2}.$$

2. (10 points) Prove the following properties of the Hilbert transform.

   (a) Let the Fourier transform of a real-valued function $f(t)$ and its Hilbert transform $\hat{f}(t)$ be denoted as $F(\omega)$ and $\hat{F}(\omega)$. Show that both $F(\omega)$ and $\hat{F}(\omega)$ exhibit Hermitian symmetry; i.e., $F(-f) = F^*(f)$ and $\hat{F}(-\omega) = \hat{F}^*(\omega)$.

   (b) Show that $g(t)$ (not necessarily real valued) and its Hilbert transform $\hat{g}(t)$ are orthogonal; i.e.,

$$< g(t), \hat{g}(t) >= \int_{-\infty}^{\infty} g(t)\hat{g}^*(t)\, dt = 0.$$

   (c) If g(t) is symmetric around $t = 0$, i.e., $g(-t) = g(t)$, then $\hat{g}(t)$ is skew-symmetric, i.e., $\hat{g}(-t) = -\hat{g}(t)$.

3. (10 points) Let $Z(t) = X(t) + jY(t)$, where $X(t)$ and $Y(t)$ are jointly wide sense stationary (WSS) and real-valued random processes. Assume that $X(t)$ and $Y(t)$ are mutually orthogonal with zero-mean functions. Define a new random process in terms of a modulation to a carrier frequency ω_0 as $U(t) = Re\{Z(t)e^{-j\omega_0 t}\}$. Given the relevant correlation functions, that is, $R_{XX}(\tau)$ and $R_{YY}(\tau)$, find general conditions on them such that $U(t)$ is also a WSS random process. Show that your conditions work, that is, the resulting process $U(t)$ is actually WSS.

4. (5 points) Let X and Y be two RVs. Show that $X \overset{\text{m.s.}}{=} Y$ (mean square) if and only if $X \overset{\text{a.s.}}{=} Y$ (almost surely).

5. (5 points) Derive the expression for the periodogram given by (13.59) in the textbook p.353.

6. (5 points) Let $\mathbf{R}$ denote the correlation matrix of a complex random vector. Prove that its eigenvectors associated with different eigenvalues are orthogonal i.e.,

$$< \mathbf{u}_i, \mathbf{u}_j >= \mathbf{u}_i^{\text{H}} \mathbf{u}_j = 0, \text{ for } i \neq j.$$

7. (10 points)

   (a) Show that the mean square error of the approximation (13.94) is given by (13.95) in the textbook p.361.

(b) Show that the first term in the final expression of (13.195) converges to X_n of (13.196) in mean square in the textbook p.386.

8. (10 points)

   (a) Show that the autoregressive sequence X_n of order 1 has the variance given by (13.201) in the textbook p.387.

   (b) Show that the autocorrelation function satisfies the recursion (13.202) and is given by (13.203) in the textbook p.387.

9. (10 points)

   (a) Let $\{X_j, j = 1, 2, ..., m\}$ be a set of independent RVs, exponentially distributed with parameters λ_j, $j = 1, 2, ..., m$, respectively. Find the distribution of the RV

$$Y = \min\{X_1, X_2, ..., X_m\}.$$

   (b) Using the result of (a), show that the superposition of m independent Poisson processes with rates $\lambda_j, j = 1, 2, ..., m$ generates a Poisson process with rate $\lambda = \sum_{i=1}^{m} \lambda_i$.

10. (5 points) Consider the problem of decomposing a Poisson stream into the m substreams, each substream receives every m-th arrival; i.e., the first arrival, $(m+1)$ arrival, $(2m+1)$ arrival,..., arrivals go to substream 1, the second, $(m+2)$, $(2m+2)$,..., arrivals go to substream 2, etc. Find the interarrival time distribution of the individual substreams.

11. (10 points) Prove the four properties of a simple random walk X_n stated in (17.3) through (17.5) in the textbook p.484.

12. (10 points) Use the properties of the Wiener process in the textbook p.492 to show the following questions:

   (a) Let Var[$W(t)$]=g(t). Show that $g(t)$ must be the form $g(t) = \alpha t$.

   (b) Show that Var[$W(t) - W(s)$]=$\alpha|t - s|$.