COM 5110
Random Processes for Communications - Spring 2018
Solutions for homework 2

1. (a)

$$M(t) = \int_{-\infty}^{0} \frac{e^{tx}\mu e^{\mu x}}{2} dx + \int_{0}^{\infty} \frac{e^{tx}\mu e^{-\mu x}}{2} dx = \frac{\mu}{2}\left\{ \left[\frac{e^{(\mu+t)x}}{\mu+t} \right]_{-\infty}^{0} + \left[\frac{e^{(t-\mu)x}}{t-\mu} \right]_{0}^{\infty} \right\}$$

$$= \begin{cases} \infty, & t \leq -\mu, \\ \frac{\mu^2}{\mu^2-t^2}, & |t| < \mu, \\ \infty, & t > \mu. \end{cases}$$

Therefore, when $|t| < \mu$, the MGF of Bilateral exponential distribution exists.

(b)

$$\begin{aligned} M(t) &= \int_{-a}^{0} \frac{e^{tx}}{a}(1+\frac{x}{a})dx + \int_{0}^{a} \frac{e^{tx}}{a}(1-\frac{x}{a})dx \\ &= \int_{-a}^{0} \frac{e^{tx}}{a} + \frac{xe^{tx}}{a^2}dx + \int_{0}^{a} \frac{e^{tx}}{a} - \frac{xe^{tx}}{a^2}dx \\ &= \int_{-a}^{0} \frac{e^{tx}}{a} + \int_{0}^{a} \frac{e^{tx}}{a} + \frac{1}{a^2}\left(\int_{-a}^{0} xe^{tx}dx - \int_{0}^{a} xe^{tx}dx \right) \\ &= \left[\frac{e^{tx}}{at} \right]_{-a}^{0} + \left[\frac{e^{tx}}{at} \right]_{0}^{a} + \frac{1}{a^2}\left(\int_{-a}^{0} xe^{tx}dx - \int_{0}^{a} xe^{tx}dx \right). \end{aligned}$$

By using the following integration by parts:

$$\int xe^{tx}dx = \frac{(tx-1)e^{tx}}{t^2}.$$

Then, substituting this into the above definite integrals, and after some manipulation, we obtain:

$$M(t) = \frac{e^{at}+e^{-at}-2}{a^2t^2} = \left(\frac{e^{\frac{at}{2}} - e^{\frac{-at}{2}}}{at} \right)^2 = \left(\frac{\sinh\frac{at}{2}}{\frac{at}{2}} \right)^2 \geq 0, \, -\infty < t < \infty.$$

2. (a) When $n_1 = 1$, $V_1 = U^2$, where U is a standard normal variable. Then

$$F_{1,n_2} = \frac{U^2}{\frac{V_2}{n_2}} = \left(\frac{U}{\sqrt{\frac{\chi^2_{n_2}}{n_2}}} \right)^2 = t^2_{n_2}.$$

Therefore, the F variable is simply the squre of the t variable of degree n_2. We can derive the distribution of F_{1,n_2} from that of t_{n2}.

$$
\begin{aligned}
f_F(x)dx &= P[x < F < x + dx] \\
&= P[x < t^2 < x + dx] \\
&= P[\sqrt{x} < t < (x+dx)^{\frac{1}{2}}] + P[-\sqrt{x} > t > -(x+dx)^{\frac{1}{2}}] \\
&\text{(By Taylor series approximation: } (1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \approx 1 + \frac{1}{2}x) \\
&= P[\sqrt{x} < t < \sqrt{x} + \frac{dx}{2\sqrt{x}}] + P[-\sqrt{x} > t > -\sqrt{x} - \frac{dx}{2\sqrt{x}}] \\
&= \frac{f_t(\sqrt{x}) + f_t(-\sqrt{x})}{2\sqrt{x}} dx \\
&= \frac{\Gamma(\frac{n_2+1}{2})}{\Gamma(\frac{n_2}{2})\sqrt{\pi n_2}} (1 + \frac{x}{n_2})^{-\frac{n_2+1}{2}} \frac{dx}{\sqrt{x}}.
\end{aligned}
$$

This is a special case of (7.40) in the textbook.

(b) The median of the distribution (7.46) in the textbook is $y_{median} = \mu_Y$ since $Y = \ln X$ is a monotone increasing function. The corresponding x_{median} is

$$
\begin{aligned}
x_{median} &= e^{y_{median}} = e^{\mu_Y} \\
&= e^{\ln \mu_X - 0.5 \ln(1 + \frac{\sigma_X^2}{\mu_X^2})} \quad \text{(From eq.(7.52) in the textbook)} \\
&= \mu_X (1 + \frac{\sigma_X^2}{\mu_X^2})^{-0.5} = \frac{\mu_X}{\sqrt{(1 + \frac{\sigma_X^2}{\mu_X^2})}}.
\end{aligned}
$$

(c) Take the logarithm of (7.47) in the textbook:

$$
\ln f_X(x) = -\frac{1}{2}\ln(2\pi) - \ln x - \frac{(\ln x - \mu_Y)^2}{2\sigma_Y^2}.
$$

Differentiate the above equation with respect to x:

$$
\frac{f_X'(x)}{f_X(x)} = -\frac{1}{x} - \frac{\ln x - \mu_Y}{\sigma_Y^2 x}.
$$

The mode x_{mode} is such x that maximizes $f_X(x)$, and thus $f_X'(x_{mode}) = 0$, Thus, we have

$$
-1 - \frac{\ln x_{mode} - \mu_Y}{\sigma_Y^2} = 0,
$$

Therefore, we obtain:

$$
x_{mode} = e^{\mu_Y - \sigma_Y^2} = \frac{x_{median}}{e^{\sigma_Y^2}} = \frac{\mu_X}{(1 + \frac{\sigma_X^2}{\mu_X^2})^{\frac{3}{2}}}.
$$

3. (a) Since -log$x$ is a convex function and by Jensen's inequality, we have

$$
E[-\log X] \geq -\log \mathrm{E}[X].
$$

Assume the RV X that takes on values $x_i = \frac{g_i}{f_i}$ with probability f_i. Then

$$
-\sum_{i=1}^{n} f_i \log \frac{g_i}{f_i} \geq -\log \sum_{i=1}^{n} f_i \frac{g_i}{f_i} = -\log 1 = 0.
$$

(b) Since $\mathbf{f}$ is fixed, it can $-f_i \log g_i$ is a convex function in g_i. Hence,

$$f(\mathbf{g}) \triangleq \sum_{i=1}^{n} -f_i \log g_i$$

is convex in $\mathbf{g}$. Instead, we now solve the following convex minimization problem:

$$\min_{\mathbf{g}} \left\{ f(\mathbf{g}) \mid \sum_{i=1}^{n} g_i = 1 \right\}$$

by solving the associated KKT conditions. The Lagrangian is given by

$$\mathcal{L}(\mathbf{g}, \lambda) = f(\mathbf{g}) + \lambda(\sum_{i=1}^{n} g_i - 1).$$

The KKT conditions are as follows:

$$\nabla_{\mathbf{g}} \mathcal{L}(\mathbf{g}, \lambda) = -[f_1/g_1, ..., f_n/g_n]^T + \lambda[1, ..., 1]^T = 0$$
$$\sum_{i=1}^{n} g_i = 1$$

By solving the above two equations, we come up with the optimal $\mathbf{g}^\star = \mathbf{f}$. Therefore,

$$f(\mathbf{g}) = \sum_{i=1}^{n} -f_i \log g_i \geq f(\mathbf{g}^\star) = \sum_{i=1}^{n} -f_i \log f_i.$$

Thus the proof has been finished.

4.

$$\bar{X}_n - c = \frac{1}{n} \sum_{k=1}^{n} (c + N_k) - c = \frac{1}{n} \sum_{k=1}^{n} N_k.$$

By the weak law of large numbers,

$$\frac{1}{n} \sum_{k=1}^{n} N_k \xrightarrow{P} \frac{1}{n} \mathrm{E}[\sum_{k=1}^{n} N_k] = 0.$$

Therefore,

$$\lim_{n \to \infty} P[|\bar{X}_n - c| \geq \epsilon] = 0,$$

for any $\epsilon > 0$. Hence, $\bar{X}_n \xrightarrow{P} c$.

5. (a) Consider Markov's inequality:

$$P[T_n \geq n\beta] \leq \frac{\mathrm{E}[T_n]}{n\beta} = \frac{n/2}{n\beta} = \frac{1}{2\beta}.$$

Therefore, for $\beta = 0.8$, the upper bound is $\frac{1}{1.6} = 0.625$ for all n.

(b) Writing $b = n\beta = \frac{n}{2} + (\beta - \frac{1}{2})n =E[T_n] + (\beta - \frac{1}{2})n$, we obtain

$$P[T_n \geq n\beta] = P[T_n - \mathrm{E}[T_n] \geq (\beta - \frac{1}{2})n] \leq P[|T_n - \mathrm{E}[T_n]| \geq (\beta - \frac{1}{2})n]$$
$$\leq \frac{\sigma^2_{T_n}}{[(\beta - \frac{1}{2})n]^2} = \frac{n/4}{(\beta - \frac{1}{2})^2 n^2} = \frac{1}{4n(\beta - \frac{1}{2})^2}, \ \frac{1}{2} < \beta < 1.$$

Thus, the probability that T_n exceeds $b = n\beta$ approaches zero as n increases. For $\beta = 0.8$, the upper bound is $\frac{1}{0.36n}$, which is 2.78×10^{-2} for $n = 100$, and 2.78×10^{-3} for $n = 1000$.

6. (a) Let $b > -a$, hence

$$P[X \geq a] = P[X + b \geq a + b] = P[(X + b)^2 \geq (a + b)^2]$$
$$\leq \frac{\mathrm{E}[(X + b)^2]}{(a + b)^2} = \frac{\sigma^2 + b^2}{(a + b)^2}.$$

Let $f(b) = \frac{\sigma^2 + b^2}{(a+b)^2}$. Then it can be shown that under if $-a < b \leq \frac{3\sigma^2 + a^2}{2a}$,

$$\nabla^2_b f(b) = (a + b)^{-4}(6\sigma^2 - 4ab + 2a^2) \geq 0$$

implying that $f(b)$ is convex over $[-a, \frac{3\sigma^2+a^2}{2a}]$. Therefore, letting $\partial f(b)/\partial b = 0$ yields $b^\star = \frac{\sigma^2}{a} \in [-a, \frac{3\sigma^2+a^2}{2a}]$. The optimal value $f(b^\star) = \frac{\sigma^2}{\sigma^2+a^2}$, implying that

$$P[X \geq a] \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

(b) $X - \mu$ has mean zero and variance σ^2, we obtain the first inequality from (a). Similarly, $\mu - X$ has mean zero and variance σ^2 and we obtain the second inequality.

7. (a)

$$M_U(\xi) = e^{\xi^2/2}, \ m_U(\xi) = \xi^2/2, \ m_U^{'}(\xi) = \xi$$

So, $\xi^\star$ that achieves a minimum is the root of

$$m_U^{'}(\xi) = \xi = \frac{b}{n} = \beta, \ i.e., \ \xi^\star = \beta.$$

$$P[S \geq b] \leq e^{-n[\xi^\star m_U^{'}(\xi^\star) - m_U(\xi^\star)]} = e^{-n(\beta^2 - \beta^2/2)} = e^{-n\beta^2/2}, \ b \geq 0.$$

Substitute $b = n\beta$ to above equation, we have finished the proof.

8. The distribution function of Z_n is given by

$$F_{Z_n}(z) = P[Z_n \le z] = P[n(1-Y_n) \le z] = P[Y_n \ge 1 - \frac{z}{n}].$$

Since $Y_n = \max\{X_1, X_2, \ldots, X_n\}$,

$$P[Y_n \le 1 - \frac{z}{n}] = P[X_i \le 1 - \frac{z}{n},\ 1 \le i \le n] = [F_X(1 - \frac{z}{n})]^n.$$

When $n > z > 0$,

$$0 \le 1 - \frac{z}{n} \le 1,$$

and

$$F_X(1 - \frac{z}{n}) = 1 - \frac{z}{n},\ n > z.$$

Hence,

$$\lim_{n\to\infty} [F_X(1 - \frac{z}{n})]^n \ = \ \lim_{n\to\infty} [1 - \frac{z}{n}]^n,$$

Let $x = \frac{z}{n}$ and use the binomial theorem $\lim_{x\to 0}(1-x)^{1/x} = e^{-1}$, we have

$$\lim_{n\to\infty} [1 - \frac{z}{n}]^n = e^{-z}.$$

From the above equations, we have

$$F_{Z_n}(z) = P[Y_n \ge 1 - \frac{z}{n}] = 1 - P[Y_n \le 1 - \frac{z}{n}] = 1 - [F_X(1 - \frac{z}{n})]^n$$

Therefore, we have

$$\lim_{n\to\infty} F_{Z_n}(z) = 1 - e^{-z},\ z \ge 0,\ \text{i.e.,}\ Z_n \xrightarrow{D} Z.$$

9. (a)

$$E[|X_n - 0|^r] = E[|X_n|^r] = \frac{1}{n} \cdot (n^{1/2r})^r + \frac{n-1}{n} \cdot 0^r = \frac{1}{n^{1/2}} \to 0, \text{as}\ n \to \infty.$$

Therefore, $X_n \xrightarrow{r} 0$.

(b) Let M be an integer such that

$$M^{1/2r} > \epsilon \implies M > \epsilon^{2r}.$$

Then, for $n \ge M$,

$$P[A_n(\epsilon)] = P[X_n < \epsilon] = P[X_n = 0] = \frac{n-1}{n}.$$

Hence, for $m \ge M$,

$$\begin{aligned} P[B_m(\epsilon)] &= P[\bigcap_{n=m}^{\infty} A_n(\epsilon)] = \prod_{n=m}^{\infty} P[A_n(\epsilon)] \\ &= \lim_{M\to\infty} \frac{m-1}{m} \cdot \frac{m}{m+1} \cdots \frac{M-1}{M} \cdot \frac{M}{M+1} \\ &= \lim_{M\to\infty} \frac{m-1}{M} = 0. \end{aligned}$$

(c) $X_n \stackrel{a.s.}{\to} 0$ if and only if $\lim_{m\to\infty} P[B_m(\epsilon)] = 1$. By the result of (b), $\{X_n\}$ does not converge almost surely to 0.

10. The normalized average Z_n is given by

$$Z_n = \sum_{k=1}^{n} \tilde{X}_k,$$

where

$$\tilde{X_k} = \frac{X_k - \mu}{\sigma\sqrt{n}}.$$

The moment generating function (MGF) of Z_n is then given by

$$\begin{aligned} M_{Z_n}(t) &= \mathrm{E}[\exp\{t\sum_{k=1}^{n}(\frac{X_k-\mu}{\sigma\sqrt{n}})\}] \\ &= \prod_{k=1}^{n} \mathrm{E}[\exp\{t(\frac{X_k-\mu}{\sigma\sqrt{n}})\}] = [M_{\tilde{X}(t)}]^n, \end{aligned}$$

where $M_{\tilde{X}}(t)$ is the common MGF of $\tilde{X_k}$. Using the Taylor series expansion, we can write $M_{\tilde{X}}(t)$ as follows:

$$\begin{aligned} M_{\tilde{X}(t)} &= 1 + \mathrm{E}[\frac{X_k-\mu}{\sigma\sqrt{n}}]t + \frac{1}{2}\mathrm{E}[(\frac{X_k-\mu}{\sigma\sqrt{n}})^2]t^2 + o(\frac{t^2}{n}) \\ &= 1 + \frac{t^2}{2n} + o(\frac{t^2}{n}) \end{aligned}$$

where $o(\frac{t^2}{n})$ denotes the sum of all the higher order terms. Hence, we have

$$\begin{aligned} \lim_{n\to\infty} M_{Z_n}(t) &= \lim_{n\to\infty}[1 + \frac{t^2}{2n} + o(\frac{t^2}{n})]^n \\ &= \lim_{n\to\infty}[1 + \frac{t^2}{2n}]^n = e^{\frac{t^2}{2}} \quad (\text{by } \lim_{x\to 0}(1+x)^{1/x} = e). \end{aligned}$$

which is the MGF of the unit normal distribution. This establishes that Z_n converges in distribution to the unit normal variable.