COM 5110
Random Processes for Communications - Spring 2018
Homework 2
(Due date: April 26, 2018)

1. (10 points)

   (a) Find the moment generating function (MGF) of a bilateral exponential distribution defined by
   $$f_X(x) = \frac{\mu}{2} e^{-\mu|x|}, \ -\infty < x < \infty,$$
   and under what condition does the MGF exist?

   (b) Find the MGF of a triangular distribution defined by
   $$f_X(x) = \frac{1}{a}(1 - \frac{|x|}{a}), \ |x| < a.$$

2. (10 points)

   (a) Show that when the degree of freedom in the numerator of F-distribution is equal to one, the F-distribution and t-distribution have a relationship.

   (b) Show that the median of the log-normal distribution $F_X(x)$ is given by:
   $$e^{\mu_Y} = \frac{\mu_X}{\sqrt{1 + \frac{\sigma_X^2}{\mu_X^2}}},$$
   where $Y = \ln X$ has the normal distribution $N(\mu_Y, \sigma_Y^2)$.

   (c) Show that the mode of the log-normal distribution $F_X(x)$ is given by:
   $$x_{mode} = e^{\mu_Y - \sigma_Y^2} = \frac{\mu_X}{(1 + \frac{\sigma_X^2}{\mu_X^2})^{\frac{3}{2}}}.$$

3. (15 points)
   $$\sum_{i=1}^{n} f_i \log g_i \le \sum_{i=1}^{n} f_i \log f_i$$
   where $\mathbf{f} = [f_1, f_2, \ldots, f_n]$ and $\mathbf{g} = [g_1, g_2, \ldots, g_n]$ are two probability distributions.

   (a) Use Jensen's inequality.

   (b) Let $\mathbf{f}$ be fixed. Prove the inequality by solving the following maximization problem
   $$\max_{\mathbf{g}} \left\{ \sum_{i=1}^{n} f_i \log g_i \mid \sum_{i=1}^{n} g_i = 1 \right\}$$

4. (5 points) Define a random variable X_k by

$$X_k = c + N_k,\ k = 1, 2, \ldots,$$

where c is a constant and the N_k are i.i.d RVs with zero mean. Show that the sequence of sample averages $\{\bar{X}_n\}$

$$\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k,\ n = 1, 2, \ldots,$$

converges in probability to c.

5. (10 points) Consider the experiment of tossing a fair coin n times. Let T_n be the total number of "head (=1)":

$$T_n = B_1 + B_2 + \cdots + B_n,$$

where B_i are independent binary variables, with $P[B_i = 1] = P[B_i = 0] = \frac{1}{2}$ for all $i = 1, 2, \ldots, n$. Assume the threshold value $b > E[T_n] = \frac{n}{2}$; i.e., $\frac{1}{2} < \beta < 1$, where $\beta = \frac{b}{n}$.

   (a) Use Markov's inequality and find an upper bound on the probability that T_n exceeds $b = \beta n$. Determine the upper bound for cases $n = 100$ and $n = 1000$ with $\beta = 0.8$ for both cases.

   (b) Repeat (a) by using Chebyshev's inequality.

6. (10 points)

   (a) Let X be a random variable with $\mathrm{E}[X] = 0$ and $\mathrm{Var}[X] = \sigma^2$. For any $a > 0$, show that

$$P[X \geq a] \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

   (b) If $\mathrm{E}[X] = \mu$ and $\mathrm{Var}[X] = \sigma^2$. Show that

$$P[X \geq \mu + a] \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

$$P[X \leq \mu - a] \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

7. (10 points) Let S be a sum of n independent standard normal variables U_i; i.e., $U_i \sim N(0, 1)$:

$$S = U_1 + U_2 + \cdots + U_n.$$

Use Chernoff's bound to show that $P[S \geq n\beta] \leq e^{\frac{-n\beta^2}{2}}$.

8. (10 points) Let X_n be a sequence of i.i.d RVs with common distribution function $F_X(x)$ given by

$$F_X(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x, & \text{if } 0 < x \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$$

Now, we define two sequences Y_n and Z_n by

$$Y_n = \max\{X_1, X_2, \ldots, X_n\}$$

and

$$Z_n = n(1 - Y_n).$$

Show that Z_n converges in distribution to a random variable Z with distribution

$$F_Z(z) = 1 - e^{-z}.$$

9. (10 points) Consider the following example to illustrate that convergence in rth mean does not imply almost surely convergence. Let X_n be a sequence of independent RVs defined by

$$X_n = \begin{cases} n^{\frac{1}{2r}}, & \text{with probability } \frac{1}{n}, \\ 0, & \text{with probability } 1 - \frac{1}{n}. \end{cases}$$

and let X be the degenerate RV

$$X = 0, \text{with probability one.}$$

(a) Show that X_n converges in the rth mean to $X = 0$.

(b) Show that for $\epsilon > 0$ and for arbitrary integer $m > 0$,

$$P[\bigcap_{n=m}^{\infty} \{\omega : X_n(\omega) < \epsilon\}] = 0.$$

(c) Show that X_n does not converge almost surely to $X = 0$.

10. (10 points) Prove the central limit theory (CLT) for i.i.d RVs (Theorem 11.22 in the textbook) by using the moment generating function (MGF) of the normalized average Z_n.