COM 5110
Random Processes for Communications - Spring 2018
Solutions for homework 1

1. The size of the sample space in this case is $|\Omega| = 365^r$. We are interested in the event A that r people have different birthdays. Hence,

$$|A| = 365 \times 364 \times \cdots \times (365 - r + 1),$$

so, we have:

$$P[A] = \frac{365 \times 364 \times \cdots \times (365 - r + 1)}{365^n} = (\frac{364}{365})(\frac{363}{365}) \dots (\frac{365 - r + 1}{365})$$
$$= (1 - \frac{1}{365})(1 - \frac{2}{365}) \dots (1 - \frac{r-1}{365}).$$

Taking the natural logarithms, and using the hint $ln(1 - x) \approx -x$, for $|x| \ll 1$, we have:

$$lnP[A] \approx -\frac{1}{365} - \frac{2}{365} - \cdots - \frac{r-1}{365} = -\frac{1}{365} \cdot \frac{r(r-1)}{2} = \frac{-r(r-1)}{730}.$$

Hence,

$$P[A] \approx exp(\frac{-r(r-1)}{730}).$$

For $r = 23$, $P[A] \approx 0.500$ and for $r = 56$, $P[A] \approx 0.0147$.

2. (a)

$$E[X] = \sum_{k=1}^{\infty} kq^{k-1}p = p \sum_{k=1}^{\infty} kq^{k-1}.$$

Then, use the formula for a geometric series:

$$\sum_{k=1}^{\infty} q^k = \frac{q}{1-q} \text{ for } |q| < 1.$$

Differentiating both sides, we have:

$$\sum_{k=1}^{\infty} kq^{k-1} = \frac{1}{(1-q)^2}$$
$$\Longrightarrow E[X] = p\frac{1}{(1-q)^2} = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

Then, we write

$$E[X^2] = E[X(X-1) + X] = \sum_{k=1}^{\infty} (k(k-1) + k)q^{k-1}p$$
$$= pq \sum_{k=1}^{\infty} k(k-1)q^{k-2} + p \sum_{k=1}^{\infty} kq^{k-1}$$

By differentiating the formula for a geometric series twice, we have:

$$\sum_{k=1}^{\infty} k(k-1)q^{k-2} = \frac{2}{(1-q)^3}$$
$$\implies E[X^2] = pq\frac{2}{(1-q)^3} + \frac{p}{(1-q)^2} = \frac{1+q}{p^2}$$
$$\implies Var[X] = E[X^2] - E[X]^2 = \frac{q}{p^2}.$$

(b)

$$E[X] = \sum_{k=0}^{\infty} k\frac{\lambda^k}{k!}e^{-\lambda} = \lambda\sum_{k=1}^{\infty}\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^i}{i!} = \lambda e^{-\lambda}e^{\lambda} = \lambda.$$

$$E[X^2] = \sum_{k=0}^{\infty} k^2\frac{\lambda^k}{k!}e^{-\lambda} = \lambda\sum_{k=1}^{\infty} k\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!} = \lambda\sum_{k=0}^{\infty}\frac{(i+1)e^{-\lambda}\lambda^i}{i!}$$
$$= \lambda(\lambda\sum_{k=0}^{\infty}\frac{ie^{-\lambda}\lambda^i}{i!} + \lambda\sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^i}{i!}) = \lambda(\lambda+1).$$

So

$$Var[X] = E[X^2] - E[X]^2 = \lambda.$$

3. (a)

$$F_{XY}(x,y) = \int_0^y\int_0^x f_{XY}(u,v)dudv = k\int_0^y\int_0^x e^{-\lambda u-\mu v}dudv$$
$$= k(\int_0^y e^{-\mu v}dv)(\int_0^x e^{-\lambda u}du) = \frac{k}{\mu\lambda}(1-e^{-\mu y})(1-e^{-\lambda x}).$$

Since $\lim_{x,y\to\infty}F_{XY}(x,y) = 1$, we find that $k = \mu\lambda$. Therefore,

$$F_{XY}(x,y) = (1-e^{-\mu y})(1-e^{-\lambda x}).$$

(b)

$$F_X(x) = \lim_{y\to\infty}F_{XY}(x,y) = 1-e^{-\lambda x}.$$

$$F_Y(y) = \lim_{x\to\infty}F_{XY}(x,y) = 1-e^{-\mu y}.$$

$$F_{Y|X}(y|x) = \frac{\frac{\partial}{\partial x}F_{XY}(x,y)}{f_X(x)} = \frac{\lambda e^{-\lambda x}(1-e^{-\mu y})}{\lambda e^{-\lambda x}} = 1-e^{-\mu y} = F_Y(y).$$

4. (a) Let X_i's be random variables. Then,

$$E\left[\sum_i a_i X_i|Y\right] = \int\int\ldots\int\left(\sum_{i=1}^{n} a_i x_i\right) f_{X_1X_2\ldots X_n|Y}(x_1,x_2,\ldots,x_n)dx_1dx_2\ldots dx_n$$
$$= \sum_{i=1}^{n} a_i \int x_i f_{X_i|Y}(x_i)dx_i = \sum_{i=1}^{n} a_i E[X_i|Y],$$

for all n and scalars $a_1, a_2, \ldots, a_n$. Thus $E[\cdot|Y]$ is a linear operator.

(b)

$$E[X|Y] = \int_x x f_{X|Y}(x)dx,$$

and

$$E[E[X|Y]] = \int_y f_Y(y)dy \int_x x f_{X|Y}(x)dx = \int_x x(\int_y f_{XY}(x,y)dy)dx = \int_x x f_X(x)dx = E[X].$$

(c)

$$E[h(Y)g(X)|Y] = \int h(Y)g(X)f_{X|Y}(x)dx = h(Y)\int g(x)f_{X|Y}(x)dx = h(Y)E[g(X)|Y].$$

5. (a) Since $f_X(z-y)$ is nonzero for $y < z$ and $f_Y(y)$ is nonzero for $y \geq 0$, we have:

$$f_Z(z) = \int_0^z f_X(z-y)f_Y(y)dy = \lambda\mu e^{-\lambda z}\int_0^z e^{-(\mu-\lambda)y}dy = \lambda\mu e^{-\lambda z}\frac{1-e^{-(\mu-\lambda)z}}{(\mu-\lambda)z}$$
$$= \frac{\lambda\mu}{\mu-\lambda}(e^{-\lambda z} - e^{-\mu z}), \text{ for } \mu \neq \lambda.$$

(b) When $\mu = \lambda \implies \mu - \lambda = 0$, we have:

$$f_Z(z) = \int_0^z f_X(z-y)f_Y(y)dy = \lambda\mu e^{-\lambda z}\int_0^z e^{-(\mu-\lambda)y}dy = \lambda\mu e^{-\lambda z}\int_0^z 1dy = \lambda\mu z e^{-\lambda z}.$$

6. (a)

$$D_z = \{(x,y) : \min(x,y) \leq z\}$$

Draw a vertical line $X = z$ and a horizontal line $Y = z$. Then the left side of $X = z$, and the area below $Y = z$ should constitute the shaded area D_z.

(b)

$$F_Z(z) = P[\min(X,Y) \leq z] = P[X \leq z \text{ or } Y \leq z] = F_X(z) + F_Y(z) - F_{XY}(z,z).$$

(c) If X and Y are independent, then

$$F_{XY}(z,z) = F_X(z)F_Y(z).$$

Then, differentiating the both side with respect to z, we have:

$$f_{XY}(z,z) = f_X(z)F_Y(z) + F_X(z)f_Y(z).$$

Therefore,

$$f_Z(z) = f_X(z) + f_Y(z) - f_X(z)F_Y(z) - F_X(z)f_Y(z).$$

(d)

$$\begin{aligned} &f_X(z) = \lambda e^{-\lambda z}, f_Y(z) = \mu e^{-\mu z}, \\ &F_X(z) = P[X \leq z] = \int_0^z f_X(a)\ da = \int_0^z \lambda e^{-\lambda a}\ da = 1 - e^{-\lambda z}, \\ &F_Y(z) = P[Y \leq z] = \int_0^z f_Y(b)\ db = \int_0^z \mu e^{-\mu b}\ db = 1 - e^{-\mu z}. \end{aligned}$$

Then, we find

$$f_Z(z) = f_X(z) + f_Y(z) - f_X(z)F_Y(z) - F_X(z)f_Y(z) = (\lambda + \mu)e^{-(\lambda+\mu)z}.$$

7. (a) Let $X_1 = R\cos\Theta, \mathrm{X}_2 = \mathrm{R}\sin\Theta$. The joint PDF of X_1, X_2 is given as

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi}e^{-\frac{x_1^2+x_2^2}{2}}.$$

The PDF of R, Θ can be found as

$$f_{R\Theta}(r, \theta) = |J| f_{X_1X_2}(x_1, x_2),$$

where

$$J = \frac{\partial(x_1, x_2)}{\partial(r, \theta)} = det(\begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}) = r.$$

Thus,

$$f_{R\Theta}(r, \theta) = \frac{r}{2\pi}e^{-\frac{r^2}{2}}.$$

Since this joint PDF does not depend on θ, the RVs R and Θ are not only independent but also Θ is uniform. Thus the joint PDF can be written as $f_\Theta(\theta)f_R(r)$, where

$$f_\Theta(\theta) = \frac{1}{2\pi},\ 0 \leq \theta \leq 2\pi,$$

and

$$f_R(r) = re^{-\frac{r^2}{2}}.$$

and the RV Θ is uniform in $[0, 2\pi]$.

(b) By integrating the PDF obtained above, we have:

$$F_R(r) = \int_0^r f_R(s)ds = 1 - e^{-\frac{r^2}{2}}.$$

8. (a)

$$f_{\chi_n^2}(v)dv = \frac{v^{\frac{n}{2}-1}e^{-\frac{v}{2}}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})},\ 0 \leq v < \infty.$$

$$\begin{aligned} &E[(\chi_n^2)^m] = \int_0^\infty v^m f_{\chi_n^2}(v)dv = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\int_0^\infty v^{m+\frac{n}{2}-1}e^{-\frac{v}{2}}dv \\ &= \frac{2^{m+\frac{n}{2}}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\int_0^\infty t^{m+\frac{n}{2}-1}e^{-t}dt = \frac{2^m\Gamma(\frac{n}{2}+m)}{\Gamma(\frac{n}{2})}. \end{aligned}$$

(b) Since $F = \frac{V_1/n_1}{V_2/n_2}$, the rth moment of F is

$$E[F^r] = (\frac{n_2}{n_1})^r E[V_1^r] E[V_2^{-r}].$$

By the result of $8(a)$, we have:

$$E[V_1^r] = \frac{2^r \Gamma(\frac{n_1}{2} + r)}{\Gamma(\frac{n_1}{2})} \text{ and } E[V_2^{-r}] = \frac{2^{-r} \Gamma(\frac{n_2}{2} - r)}{\Gamma(\frac{n_2}{2})}.$$

Hence,

$$E[F^r] = (\frac{n_2}{n_1})^r \frac{\Gamma(\frac{n_1}{2} + r)\Gamma(\frac{n_2}{2} - r)}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})}.$$

From the conditions $\frac{n_1}{2} + r > 0$ and $\frac{n_2}{2} - r > 0$, we obtain $-n_1 < 2r < n_2$.

9. (a)

$$M(t) = E[e^{tX}] = \int_0^a \frac{e^{tx}}{a} dx = \left[\frac{e^{tx}}{at}\right]_0^a = \frac{e^{at} - 1}{at}, \text{ for } t \neq 0.$$

For $t = 0$, by definition $M(0) = E[e^0] = 1$.

(b)

$$M(t) = \frac{1}{2a} \int_{-a}^a e^{tx} dx = \frac{1}{2a} \left[\frac{e^{tx}}{t}\right]_{-a}^a = \frac{e^{at} - e^{-at}}{2at} = \frac{\text{sinh(at)}}{at}, \text{ for } t \neq 0.$$

For $t = 0$, by definition $M(0) = 1$.

(c)

$$M(t) = \int_0^\infty e^{tx} \mu e^{-\mu x} dx = \mu \left[\frac{e^{(t-\mu)x}}{t - \mu}\right]_0^\infty = \begin{cases} \frac{\mu}{\mu - t}, & \text{if } t < \mu \\ \infty, & \text{if } t \geq \mu. \end{cases}$$

10. (a) By definition,

$$\begin{aligned} \phi(u) &= \sum_{k=0}^n B(k; n, p) e^{iuk} = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} e^{iuk} \\ &= \sum_{k=0}^n \binom{n}{k} (pe^{iu})^k (1-p)^{n-k} = (pe^{iu} + 1 - p)^n. \end{aligned}$$

(b)

$$\phi(u) = \sum_{k=0}^\infty \frac{\lambda^k}{k!} e^{-\lambda} e^{iuk} = e^{-\lambda} \sum_{k=0}^\infty \frac{(\lambda e^{iu})^k}{k!} = e^{-\lambda} e^{\lambda e^{iu}} = e^{\lambda(e^{iu} - 1)}.$$