COM 5110
Random Processes for Communications - Spring 2018
Homework 1
(Due date: March 26, 2018)

(Please write down the answer in detail !)

1. (10 points) In a group of r persons, what is the probability that each person has a distinct birthday? We assume that birth rates are constant throughout the year, and ignore complications due to leap years. Evaluate (approximately) this probability for $r = 23$ and $r = 56$. $Hint$: Use the approximation $ln(1-x) \approx -x$ for $|x| \ll 1$.

2. (10 points) Detail the following questions:

   (a) Show that the mean and the variance of the geometric RV X are as $E[\text{X}] = \frac{1}{p}$ and $Var[\text{X}] = \frac{q}{p^2}$ where p is the probability of success, $q = 1 - p$ is the probability of failure.

   (b) If $X \sim \text{Poisson}(\lambda)$, show that the mean, the second moment, and the variance.

3. (10 points) Consider a pair of continuous RVs (X, Y) that have a joint PDF of the form:

$$f_{XY}(x, y) = \begin{cases} ke^{-\lambda x - \mu y}, & \text{if } x \geq 0, y \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

   where $\lambda > 0,\ \mu > 0$.

   (a) Obtain the joint distribution function $F_{XY}(x, y)$ and determine the normalization constant k.

   (b) Find the distribution functions $F_X(x)$ and $F_Y(y)$ and the conditional distribution function $F_{Y|X}(y|x)$.

4. (10 points) Show the following properties of conditional expection for two continuous RVs X and Y:

   (a) $E[\cdot|Y]$ is a linear operator.

   (b) $E[E[X|Y]] = E[X]$.

   (c) $E[h(Y)g(X)|Y] = h(Y)E[g(X)|Y]$, where h and g are scalar functions.

5. (10 points) Let X and Y be independent exponential variables with rate parameters λ and μ, respectively:

$$f_X(x) = \lambda e^{-\lambda x}u(x) \text{ and } f_Y(y) = \mu e^{-\mu y}u(y),$$

   where $u(\cdot)$ is the unit step function.

   (a) Find the PDF of $Z = X + Y$.

   (b) What form does $f_Z(z)$ take when $\lambda = \mu$?

6. (10 points) Let

$$Z = \min\{X, Y\} = \begin{cases} X, & \text{if } X \le Y, \\ Y, & \text{otherwise.} \end{cases}$$

   (a) Find the domain D_z and sketch the region in X-Y plane.
   (b) Show that the distribution function of Z is given by

$$F_Z(z) = F_X(z) + F_Y(z) - F_{XY}(z, z).$$

   (c) Find the PDF $f_Z(z)$ when X and Y are independent.
   (d) Let X and Y be both exponentially distributed with rate parameters λ amd μ, respectively. Show that Z is also exponentially distributed.

7. (10 points) Let X_1 and X_2 are two independent RVs, both of which are from the normal distribution with zero mean and variance equal to one, i.e., $N(0, 1)$. Then, transform the pair (X_1, X_2) into the polar coordinates (R, Θ):

   (a) Show that the distribution function of R is given by

$$F_R(r) = 1 - e^{\frac{-r^2}{2}}$$

   and Θ is a uniform RV over $[0, 2\pi]$.
   (b) Show that $Y = X_1^2 + X_2^2$ is exponentially distributed with mean 2.

8. (10 points)

   (a) Show that the χ_n^2 has the mth moment as

$$E[(\chi_n^2)^m] = \frac{2^m \Gamma(\frac{n}{2} + m)}{\Gamma(\frac{n}{2})}, \quad m = 1, 2, 3, \ldots$$

   (b) Show that the rth moment of the F-distribution with (n_1, n_2) degree of freedom is given by

$$E[F^r] = \frac{(\frac{n_2}{n_1})^r \Gamma(\frac{n_1}{2} + r)\Gamma(\frac{n_2}{2} - r)}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})},$$

   which exists only for $-n_1 < 2r < n_2$. *Hint* : Use the result of 8(a).

9. (10 points)

   (a) Show that the moment-generating function (MGF) of the uniform distribution

$$f_X(x) = \frac{1}{a}, \quad 0 < x < a,$$

   is given by

$$M_X(t) = \begin{cases} \frac{e^{at} - 1}{at}, & \text{if } t \neq 0, \\ 1, & \text{t} = 0. \end{cases}$$

   (b) Derive the MGF of the uniform distribution

$$f_X(x) = \frac{1}{2a}, \quad |x| < a.$$

   (c) Derive the MGF of the exponential distribution

$$f_X(x) = \mu e^{-\mu x}, \quad 0 \le x < \infty.$$

10. (10 points)

(a) Show that the characteristic function (CF) of the binomial distribution $B(k; n, p), k = 0, 1, 2, \ldots, n$ is given by

$$\phi(u) = (pe^{iu} + 1 - p)^n \ , \quad -\infty < u < \infty.$$

(b) Show that the Poisson distribution with mean λ has the CF given by

$$\phi(u) = e^{\lambda(e^{iu}-1)} \ , \quad -\infty < u < \infty.$$