Let $\Omega = [0,1]$ (sample space), $\mathcal{F}$ be the σ-field generated by events $[a,b]$ for any $0 \le a \le b \le 1$ and $P\big[[a,b]\big] = b - a$ be the Lebesgue measure (also a probability measure), thereby $(\Omega, \mathcal{F}, P)$ forming a probability space. Let us consider some examples for illustrating the convergence in probability and almost sure convergence of random sequences defined on this probability space.

Let $Y_{n,m}$ for $n = 1, 2, \ldots, m = 1, \ldots, n$ be random variables defined as

$$Y_{n,m}(\omega) = \begin{cases} 1, & \omega \in A_{n,m} \triangleq \Omega \setminus [(m-1)/n, m/n] \\ 0, & \text{otherwise} \end{cases} \tag{0.1}$$

and thus $P[Y_{n,m} = 1] = 1 - 1/n$ and $P[Y_{n,m} = 0] = 1/n$ for all m. Let $X = 1$ and X_k be a random sequence defined by $Y_{n,m}$ as follows

$$\begin{aligned} &Y_{1,1} = X_1, \\ &Y_{2,1} = X_2,\ Y_{2,2} = X_3, \\ &\ \ldots \\ &Y_{n,1} = X_{n(n-1)/2+1}, \ldots, Y_{n,n} = X_{n(n+1)/2}, \\ &\ \ldots \end{aligned}$$

Then for $0 < \epsilon < 1$ we have

$$\begin{aligned} P[A_{n,m}] &= P[|X_{n(n-1)/2+m} - X| \le \epsilon] \\ &= P[X_{n(n-1)/2+m} = 1] = 1 - \frac{1}{n} \end{aligned} \tag{0.2}$$

implying that

$$\lim_{k\to\infty} P[|X_k - X| \le \epsilon] = \lim_{n\to\infty} P[|Y_{n,m} - X| \le \epsilon] = 1,$$

i.e., X_k converges to $X = 1$ in probability. However, for any $\omega^\star \in \Omega$ and any finite positive integer N, there exists at least an $A_{n,m}$ such that $\omega^\star \in A_{n,m}^C = [(m-1)/n, m/n]$ where $n(n-1)/2 + m > N$. In other words, it is impossible that $|X_k(\omega^\star) - X(\omega^\star)| \le \epsilon$ for all $k > N$, or it must be true that

$$\bigcap_{n>N} \bigcap_{m=1}^{n} A_{n,m} = \emptyset$$

implying that X_k does not converge almost surely to $X = 1$.

Consider another random sequence X_k defined as follows.

$$X_k(\omega) = \begin{cases} 1, & \omega \in A_k \triangleq [0, \sum_{n=1}^{k} (0.5)^n] \\ 0, & \text{otherwise} \end{cases} \tag{0.3}$$

and so $P[X_k = 1] = \sum_{n=1}^{k} (0.5)^n$ and $P[X_k = 0] = 1 - P[X_k = 1]$. It can be seen that $A_m \subset A_k = \{\omega : |X_k(\omega) - X(\omega)| < \epsilon\}$ for $m < k$ and $0 < \epsilon < 1$. Thus

$$\lim_{m\to\infty} \left\{ B_m \triangleq \bigcap_{k=m}^{\infty} A_k \right\} = [0,1) = \Omega \setminus \{1\}$$

implying that X_k converges to $X = 1$ with probability 1 since $P[[0, 1)] = 1$. Surely, X_k converges to $X = 1$ in probability as well since almost sure convergence implies convergence in probability.