**Convex Optimization for Communications and Signal Processing** **Fall 2020**
**Institute of Communications Engineering**
**Department of Electrical Engineering**
**National Tsing Hua University**

# Homework #3

## Coverage: Chapter 1–4

## Due date: 30 December, 2020

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**Notice**:

1. *Please hand in the hardcopy of your answer sheets to the TAs by yourself before the deadline.*
2. *Ansewrs to the problem set should be written on the A4 papers.*
3. *Write your name, student ID, and department on the begining of your ansewr sheets.*
4. *Please do the homework independently by yourself, and support your answers with clear, logical and solid reasoning or proofs.*
5. *We will grade the homework and provide the solutions afterwards. However, it is not required to hand in your homework since the percentage of homework is zero in the term grade.*

**Problem 1. (10 points)**

(a) (3 points) Consider $f : \mathbb{R}^n \to \mathbb{R}$ as a convex function and denote $\mathbf{x}^\star$ as its global minimum. For all $\mathbf{y} \in \mathbb{R}^n$, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(\alpha) \triangleq f(\mathbf{x}^\star + \alpha \mathbf{y})$ is defined. Prove that $\mathbf{x}^\star$ is the global minimum of $f$ *if and only if* $\forall \mathbf{y} \in \mathbb{R}^n$, $\alpha^\star = 0$ is the global minimum of the function $g(\alpha)$.

(b) (7 points) Let's consider a case where the function $f$ is nonconvex. Denote $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{x}^\star = (0, 0)$ and $p, q \in \mathbb{R}_{++}$, $p < q$. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = (x_2 - px_1^2)(x_2 - qx_1^2)$. Show that if $f(y, my^2) < 0$ for $y \neq 0$ and $m$ satisfying $p < m < q$, then $\mathbf{x}^\star$ is not a local minimum of $f$ even though it is a local minimum along every line passing through $\mathbf{x}^\star$.

**Problem 2. (10 points)** We consider the complex least $\ell_p$-norm problem

$$\begin{aligned} \min_{\mathbf{x}} \ & \|\mathbf{x}\|_p \\ \text{s.t. } & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{aligned}$$

where $\mathbf{A} \in \mathbb{C}^{m \times n}$, $\mathbf{b} \in \mathbb{C}^m$, and the variable is $\mathbf{x} \in \mathbb{C}^n$. Here $\| \cdot \|_p$ denotes the $\ell_p$-norm on $\mathbb{C}^n$, defined as

$$\|\mathbf{x}\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

for $p \geq 1$, and $\|\mathbf{x}\|_\infty = \max_{i=1,\ldots,n} |x_i|$. We assume that $\mathbf{A}$ is full rank, and $m < n$.

(a) (5 points) Formulate the complex least $\ell_2$-norm problem as a least $\ell_2$-norm problem with real problem data and variable.

(b) (5 points) Formulate the complex least $\ell_2$-norm problem as an SOCP.

**Problem 3. (15 points)** Consider the following optimization problem:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^\mathbf{n}} \ & \sup_{\mathbf{P} \in \mathcal{D}} (1/2)\mathbf{x}^T \mathbf{P} \mathbf{x} \\ \text{s.t. } & \mathbf{A}\mathbf{x} \preceq \mathbf{b} \end{aligned}$$

where $\mathcal{D}$ is a subset of possible matrices $\mathbf{P}$.

(a) (5 points) Let $\mathcal{D} = \{\mathbf{P}_1, \ldots, \mathbf{P}_K\} \subseteq \mathbb{S}_+^n$. Please reformulate the above problem into a QCQP using epigraph reformulation.

(b) (5 points) Let $\mathcal{D} = \{\mathbf{P} \in \mathbb{S}^n \mid -\gamma \mathbf{I} \preceq \mathbf{P} - \mathbf{P}_0 \preceq \gamma \mathbf{I}\}$, where $\gamma \in \mathbb{R}_{++}$ and $\mathbf{P}_0 \in \mathbb{S}_+^n$. Please reformulate the above problem into a QP.

(c) (5 points) Let $\mathcal{D} = \{\mathbf{P}_0 + \sum_{i=1}^{K} \mathbf{P}_i u_i \mid \|\mathbf{u}\|_2 \leq 1\}$, where $\mathbf{P}_i \in \mathbb{S}_+^n, i = 0, \ldots, K$. Please reformulate the above problem into a SOCP using epigraph reformulation.

**Problem 4. (10 points)** The function $f : \mathbb{R}^n \to \mathbb{R}$ is a quasiconvex function. Consider the convex set $X \subseteq \mathbb{R}^n$ and denote $p^\star = \inf_{\mathbf{x} \in X} f(\mathbf{x})$. Assume that $f$ is not constant on any line segment of $X$. Prove that every local minimum of $f$ over the set $X$, is also the global minimum.

**Problem 5. (15 points)** Formulate the following optimization problem as a semidefinite program (SDP).

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & \sup_{\|\mathbf{c}\|_2\le 1} \mathbf{c}^T\mathbf{F}(\mathbf{x})^{-1}\mathbf{c} \\ \text{s.t.} \quad & \mathbf{F}(\mathbf{x}) \succ \mathbf{0}, \end{aligned}$$

where

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + x_1\mathbf{F}_1 + \cdots + x_n\mathbf{F}_n$$

and each $\mathbf{F}_i \in \mathbb{S}^m$.

**Problem 6. (10 points)**

(a) (5 points) Verify that $\mathbf{x} \in \mathbb{R}^n$, $y, z, \in \mathbb{R}$ satisfy

$$\mathbf{x}^T\mathbf{x} \le yz, \;\; y \ge 0, \;\; z \ge 0$$

if and only if

$$\left\| \begin{bmatrix} 2x \\ y-z \end{bmatrix} \right\|_2 \le y+z, \;\; y \ge 0, \;\; z \ge 0.$$

(b) (5 points) Reformulate the following problem as an SOCP by (a).

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^n} \quad & \left( \sum_{i=1}^m \frac{1}{\mathbf{a}_i^T\mathbf{x} - b_i} \right)^{-1} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \succeq \mathbf{b}, \end{aligned}$$

where $\mathbf{a}_i^T$ is the $i$th row of $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $b_i$ is the $i$th element of $\mathbf{b} \in \mathbb{R}^m$.

**Problem 7. (10 points)** Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Formulate the following problems as LPs. Explain the relation between the optimal solutions of the original problem and reformulated problem.

(a) (5 points)

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_\infty.$$

(b) (5 points)

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1.$$

**Problem 8. (10 points)**

(a) (5 points) Show that $\mathbf{X} = \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$ solves the SDP

$$\begin{aligned} \min \quad & \mathrm{Tr}(\mathbf{X}) \\ \text{s.t.} \quad & \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{X} \end{bmatrix} \succeq \mathbf{0}, \end{aligned}$$

with variable $\mathbf{X} \in \mathbb{S}^n$, where $\mathbf{A} \in \mathbb{S}^m_{++}$ and $\mathbf{B} \in \mathbb{R}^{m\times n}$ are given.

(b) (5 points) Show that $\operatorname{Tr}\left(\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}\right)$ is a convex function of $(\mathbf{A}, \mathbf{B})$.

**Problem 9. (15 points)** Consider the following optimization problem:

$$\begin{aligned} \min_{x\in\mathbb{R}, y\in\mathbb{R}} \quad & f_0(x,y) \triangleq (x-2)^2 + (y-3)^2 \\ \text{s.t.} \quad & x \geq 0 \\ & x \leq 2 \\ & y \geq 0 \\ & y \leq 3 \\ & 2x + y \leq 5. \end{aligned}$$

(a) (5 points) Show that problem is a convex optimization problem.

(b) (10 points) Find an optimal solution, $(x^\star, y^\star)$. (You need to write the details of derivations.)