Convex Optimization for Communications and Signal Processing **Fall 2020**
Institute of Communications Engineering
Department of Electrical Engineering
National Tsing Hua University

Homework #2

Coverage: Chapter 1–3

Due date: 25 November, 2020

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Notice:

1. *Please hand in the hardcopy of your answer sheets to the TAs by yourself before the deadline.*
2. *Ansewrs to the problem set should be written on the A4 papers.*
3. *Write your name, student ID, and department on the begining of your ansewr sheets.*
4. *Please do the homework independently by yourself, and support your answers with clear, logical and solid reasoning or proofs.*
5. *We will grade the homework and provide the solutions afterwards. However, it is not required to hand in your homework since the percentage of homework is zero in the term grade.*

Problem 1. (10 points) Let $S \in \mathbb{R}^n$ be a compact set. Let $N : \mathbb{R}^n \to \mathbb{R}$ and $D : \mathbb{R}^n \to \mathbb{R}$ be continuous functions of $\mathbf{x}$, and $D(\mathbf{x}) > 0$ for all $\mathbf{x} \in S$. Show that the function F, defined as

$$F(t) = \max\left\{N(\mathbf{x}) - tD(\mathbf{x}) \mid \mathbf{x} \in S\right\}, \quad \mathbf{dom}\ F = \mathbb{R},$$

is convex.

Problem 2. (10 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex and differentiable, with $\mathbb{R}_+ \subseteq \mathbf{dom}\ f$. Show that the function F, defined as

$$F(x) = \frac{1}{x}\int_0^x f(t)\ dt, \quad \mathbf{dom}\ f = \mathbb{R}_{++},$$

is convex.

Problem 3. (10 points) Prove that a function $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex *if and only if* for all $\mathbf{x} \in \mathbf{dom}\, f$ and for all $\mathbf{v}$, the function $g(t) = f(\mathbf{x} + t\mathbf{v})$ is quasiconvex on its domain $\{t \mid \mathbf{x} + t\mathbf{v} \in \mathbf{dom}\, f\}$.

Problem 4. (25 points) Show that the following functions are convex/concave:

(a) (5 points) $f(\mathbf{X}) = \sup_{\mathbf{v}\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\|\mathbf{X}\mathbf{v}\|_2}{\|\mathbf{v}\|_2}$ on $\mathbb{R}^{m\times n}$.

(b) (5 points) $f(\mathbf{x}, \mathbf{Y}) = \mathbf{x}^T\mathbf{Y}^{-1}\mathbf{x}$ on $\mathbb{R}^n \times \mathbb{S}^n_{++}$.

(c) (7 points) $f(\mathbf{X}) = (\det(\mathbf{X}))^{1/m}$ with $\mathbf{dom}\ f = \mathbb{S}^m_{++}$.

(d) (8 points) $f(\mathbf{x}) = \left(\det\left(\mathbf{A}_0 + x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n\right)\right)^{1/m}$, on $\{\mathbf{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n | \ \mathbf{A}_0 + x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n \succ 0\}$, where $\mathbf{A}_i \in \mathbb{S}^m$.

Problem 5. (15 points) Let $f(\mathbf{X}) = \text{rank}(\mathbf{X})$, with $\mathbf{dom}\ f \triangleq \mathcal{B} = \{\mathbf{X} \in \mathbb{R}^{M\times N} \mid \|\mathbf{X}\|_* = \|\mathbf{X}\|_{\mathcal{A}} \leq 1\} =$ $\mathbf{conv}\ \mathcal{A}$, where $\mathcal{A}$ is the set of rank-1 matrices. Prove that the convex envelope of f can be shown to be

$$g_f(\mathbf{X}) = \|\mathbf{X}\|_* = \sum_{i=1}^{\text{rank}(\mathbf{X})} \sigma_i(\mathbf{X}), \quad \mathbf{X} \in \mathcal{B}.$$

Problem 6. (15 points) Let $K \subseteq \mathbb{R}^m$ be a proper cone with the associated generalized inequality $\preceq_K$, and let $f : \mathbb{R}^n \to \mathbb{R}^m$. For $\boldsymbol{\alpha} \in \mathbb{R}^m$, the $\boldsymbol{\alpha}$-sublevel set of f (with respect to $\preceq_K$) is defined as

$$C_{\boldsymbol{\alpha}} = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \preceq_K \boldsymbol{\alpha}\}.$$

The epigraph of f, with respect to $\preceq_K$, is defined as the set

$$\mathbf{epi}_K\ f = \{(\mathbf{x}, \mathbf{t}) \in \mathbb{R}^{n+m} \mid f(\mathbf{x}) \preceq_K \mathbf{t}\}.$$

Show the following:

(a) (4 points) If f is K-convex, then its sublevel sets $C_{\boldsymbol{\alpha}}$ are convex for all $\boldsymbol{\alpha}$.

(b) (5 points) f is K-convex if and only if $\mathbf{epi}_K\ f$ is a convex set.

(c) (6 points) $\mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$ is $\mathbb{S}^k_+$-convex on $\{(\mathbf{A}, \mathbf{B}) \in \mathbb{S}^k_{++} \times \mathbb{R}^{k\times(n-k)}\}$.

Problem 7. (15 points) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Prove the following.

(a) (10 points) If f and g are convex, both nonincreasing (or nondecreasing), and positive function on an interval, then the product of two functions fg is convex on the same interval.

(b) (5 points) If f is convex, nondecreasing, and positive, and g is concave, nonincreasing, and positive, then the ratio of two functions f/g is convex.