Convex Optimization for Communications and Signal Processing **Fall 2020**
Institute of Communications Engineering
Department of Electrical Engineering
National Tsing Hua University

Homework #1

Coverage: chapter 1–2

Due date: 28 October, 2020

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Notice:

1. *Please hand in the hardcopy of your answer sheets to the TAs by yourself before the deadline.*
2. *Ansewrs to the problem set should be written on the A4 papers.*
3. *Write your name, student ID, and department on the begining of your ansewr sheets.*
4. *Please do the homework independently by yourself, and support your answers with clear, logical and solid reasoning or proofs.*
5. *We will grade the homework and provide the solutions afterwards. However, it is not required to hand in your homework since the percentage of homework is zero in the term grade.*

Problem 1. (5 points) Let $\mathcal{F} = \{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1^2 \leq x_2\}$. Is this a convex set? Justify your answer.

Problem 2. (5 points) Let K_i for $i = 1, \ldots, n$ be cones.

(a) (2 points) Is $\mathcal{K}_1 \triangleq \cap_{i=1}^n K_i$ a cone? Justify your answer.

(b) (3 points) Let $\mathcal{K}_2 \triangleq \{\mathbf{x} = \sum_{i=1}^n \mathbf{x}_i | \mathbf{x}_i \in K_i\}$. Is $\mathcal{K}_2$ a cone? Justify your answer.

Problem 3. (10 points)

(a) (3 points) Represent the set $\left\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+ \,\middle|\, x_1 x_2 \geq 1\right\}$, as the intersection of some family of halfspaces.

(b) (3 points) Suppose that C and D are disjoint subsets of $\mathbb{R}^n$. Show that the set,

$$A = \left\{(\mathbf{a}, b) \in \mathbb{R}^{n+1} \,|\, \mathbf{a}^T\mathbf{x} \leq b \,\forall \mathbf{x} \in C, \;\; \mathbf{a}^T\mathbf{x} \geq b \,, \forall \mathbf{x} \in D\right\},$$

is a **convex cone**.

(c) (4 points) Consider two solid convex cones K_1 and K_2. Show that if $\mathbf{int}\, K_1 \cap \mathbf{int}\, K_2 = \emptyset$, then there is $\mathbf{y} \neq \mathbf{0}$ such that, $\mathbf{y} \in K_1^*$, $-\mathbf{y} \in K_2^*$.

Problem 4. (10 points) Consider the set of points, $X \triangleq \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and point $\mathbf{q}$, all in $\mathbb{R}^d$. Let $\mathcal{H}(\mathbf{q} - \mathbf{x}_i, \mathbf{x}_i)$ be the hyperplane that contains $\mathbf{x}_i$ and is perpendicular to the line segment between $\mathbf{q}$ and $\mathbf{x}_i$. Define $H_{\mathbf{q}}(\mathbf{x}_i)$ as the halfspace that does not contain the point $\mathbf{q} \in \mathbb{R}^d$ and bounded by hyperplane $\mathcal{H}(\mathbf{q} - \mathbf{x}_i, \mathbf{x}_i)$. Show that

$$\bigcap_{i=1}^n H_{\mathbf{q}}(\mathbf{x}_i) = \emptyset \quad \Leftrightarrow \quad \mathbf{q} \in \mathbf{conv}\, X.$$

Problem 5. (20 points) Which of the following sets S are polyhedra? If possible, express S in the form $S = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \preceq \mathbf{b}, \mathbf{F}\mathbf{x} = \mathbf{g}\}$.

(a) $S = \{y_1\mathbf{a}_1 + y_2\mathbf{a}_2 \mid -1 \leq y_1 \leq 1, \;\; -1 \leq y_2 \leq 1\}$, where $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^n$ are linearly independent.

(b) $S = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \succeq \mathbf{0}, \; \mathbf{1}^T\mathbf{x} = 1, \; \sum_{i=1}^n x_i a_i = b_1, \; \sum_{i=1}^n x_i a_i^2 = b_2\}$, where $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.

(c) $S = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \succeq \mathbf{0}, \; \mathbf{x}^T\mathbf{y} \leq 1$ for all $\mathbf{y}$ with $\|\mathbf{y}\|_2 = 1\}$.

(d) $S = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \succeq \mathbf{0}, \; \mathbf{x}^T\mathbf{y} \leq 1$ for all $\mathbf{y}$ with $\sum_{i=1}^n |y_i| = 1\}$.

Problem 6. (10 points) Consider the sets $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C} \subseteq \mathbb{R}^n$, where $\mathcal{A}$ is a closed set. Assume that $\mathcal{H}$ is a supporting hyperplane of $\mathcal{A}$ and $\mathcal{C}$. Prove $(\mathbf{bd}\ \mathcal{A}) \cap \mathcal{H} \subseteq \mathbf{bd}\ \mathcal{B}$.

Problem 7. (10 points) A set C is *midpoint convex* if whenever two points a, b are in C, the average or midpoint $(a+b)/2$ is in C. Obviously a convex set is midpoint convex. Prove that if C is closed and midpoint convex, then C is convex.

Problem 8. (15 points) Define the *monotone nonnegative cone* as

$$K = \{\mathbf{x} = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}.$$

(a) Show that K is a proper cone.

(b) Find the dual cone K^*. *Hint.* Use the identity

$$\begin{aligned}\sum_{i=1}^n x_i y_i = {} & (x_1 - x_2)y_1 + (x_2 - x_3)(y_1 + y_2) + (x_3 - x_4)(y_1 + y_2 + y_3) \\ & + \cdots + (x_{n-1} - x_n)(y_1 + \cdots + y_{n-1}) + x_n(y_1 + \cdots + y_n).\end{aligned}$$

Problem 9. (15 points) Let C be a set in $\mathbb{R}^n$. For any $\mathbf{x} \in \mathbf{conv}\ C$, prove that it can be represented as

$$\mathbf{x} = \sum_{i=1}^{d} \theta_i \mathbf{x}_i, \quad \theta_i \in [0,1], \ \sum_{i=1}^{d} \theta_i = 1,$$

where $\mathbf{x}_i \in C$ such that $d \leq n+1$.

Problem 10. (10 points) Suppose that C and D are closed convex cones in $\mathbb{R}^n$, and C^* and D^* are the associated dual cones. Show that

$$(C \cap D)^* = C^* + D^*.$$