Convex Optimization for Communications and Signal Processing **Fall 2018**
Institute of Communications Engineering
Department of Electrical Engineering
National Tsing Hua University

Homework #3

Coverage: chapter 1–4 & 9

Due date: 21 December, 2018

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Notice:

1. *Please hand in the hardcopy of your answer sheets to the TAs by yourself before 23:59 of the due date. No late homework will be accepted.*
2. *Ansewrs to the problem set should be written on the A4 papers.*
3. *Write your name, student ID, email and department on the begining of your ansewr sheets.*
4. *You should have a clear idea to solve a problem. Please support your answers with clear, logical and solid reasoning or proofs.*
5. *Please do the homework independently by yourself. However, you may discuss with someone else but copyied homework is not allowed. This will show your respect toward the academic integrity.*
6. *Your legible handwriting is fine. However, you are very welcome to use text formatting packages (e.x., LaTeX) for writing your answers.*
7. *Please write your best solution for each problem. There will be bonus for the creative and beautiful solutions!*

Problem 1. (10 points) Let $f(\mathbf{x}) = -x_1 x_2$, with $\mathbf{dom}\ f = \mathbb{R}^2_{++}$. Verify quasiconvexity of this function using

(a) (5 points) first-order condition for quasiconvexity.

(b) (5 points) second-order condition for quasiconvexity.

Problem 2. (10 points)

(a) (5 points) Verify that $\mathbf{x} \in \mathbb{R}^n$, $y, z, \in \mathbb{R}$ satisfy

$$\mathbf{x}^T\mathbf{x} \leq yz,\ \ y \geq 0,\ \ z \geq 0$$

if and only if

$$\left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2 \leq y + z,\ \ y \geq 0,\ \ z \geq 0.$$

(b) (5 points) Reformulate the following problem as an SOCP by (a).

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^n} \quad & \left(\sum_{i=1}^{m} \frac{1}{\mathbf{a}_i^T\mathbf{x} - b_i} \right)^{-1} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \succeq \mathbf{b}, \end{aligned}$$

where $\mathbf{a}_i^T$ is the ith row of $\mathbf{A} \in \mathbb{R}^{m\times n}$ and b_i is the ith element of $\mathbf{b} \in \mathbb{R}^m$.

Problem 3. (10 points) We consider the complex least ℓ_p-norm problem

$$\begin{aligned} \min_{\mathbf{x}} \ & \|\mathbf{x}\|_p \\ \text{s.t.}\ & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{aligned}$$

where $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{b} \in \mathbb{C}^m$, and the variable is $\mathbf{x} \in \mathbb{C}^n$. Here $\|\cdot\|_p$ denotes the ℓ_p-norm on $\mathbb{C}^n$, defined as

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

for $p \geq 1$, and $\|\mathbf{x}\|_\infty = \max_{i=1,\ldots,n} |x_i|$. We assume that $\mathbf{A}$ is full rank, and $m < n$.

(a) (5 points) Formulate the complex least ℓ_2-norm problem as a least ℓ_2-norm problem with real variable.

(b) (5 points) Formulate the complex least ℓ_2-norm problem as an SOCP.

Problem 4. (10 points) Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Formulate the following problems as LPs. Explain the relation between the optimal solutions of the original problem and reformulated problem.

(a) (5 points)

$$\min_{\mathbf{x}\in\mathbb{R}^n} \ \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_\infty.$$

(b) (5 points)

$$\min_{\mathbf{x}\in\mathbb{R}^n} \ \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1.$$

Problem 5. (20 points) Consider the convex piecewise-linear minimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \ \max_{i=1,\ldots,m} (\mathbf{a}_i^T\mathbf{x} + b_i) \tag{1}$$

(a) (5 points) Derive the Lagrange dual problem of the following epigraph reformulation of (1)

$$\min_{\mathbf{x}\in\mathbb{R}^n, y\in\mathbb{R}} \quad y \tag{2a}$$

$$\text{s.t. } \mathbf{A}^T\mathbf{x} + \mathbf{b} \preceq y\mathbf{1}_m, \tag{2b}$$

where $\mathbf{1}_m$ is the $m \times 1$ all-one vector.

(b) (5 points) Suppose we approximate the nonsmooth objective function in (1) by the smooth function

$$f_0(\mathbf{x}) = \ln\left(\sum_{i=1}^{m} \exp(\mathbf{a}_i^T\mathbf{x} + b_i)\right),$$

and obtain the following convex problem

$$\min_{\mathbf{x}\in\mathbb{R}^n, \mathbf{y}\in\mathbf{R}^m} \quad \ln\left(\sum_{i=1}^{m} e^{y_i}\right) \tag{3a}$$

$$\text{s.t. } \mathbf{A}^T\mathbf{x} + \mathbf{b} \preceq \mathbf{y}. \tag{3b}$$

Derive the Lagrange dual problem of (3).

(c) (5 points) Let $p_\ell^\star$ and $p_g^\star$ denote the optimal values of (2) and (3), respectively. Show that

$$0 \le p_g^\star - p_\ell^\star \le \ln m.$$

(d) (5 points) Derive a similar bound for the difference between $p_\ell^\star$ and the optimal value of

$$\min_{\mathbf{x}\in\mathbb{R}^n, \mathbf{y}\in\mathbb{R}^m} \quad \frac{1}{\gamma}\ln\left(\sum_{i=1}^{m} e^{\gamma y_i}\right)$$

$$\text{s.t. } \mathbf{A}^T\mathbf{x} + \mathbf{b} \preceq \mathbf{y},$$

where $\gamma > 0$ is a parameter. What happens as we increase γ?

Problem 6. (20 points) Consider the following problem. Given k points $\mathbf{v}_1, \ldots, \mathbf{v}_k$ belonging to $\mathbb{R}^n$.

(a) (5 points) Formulate a problem such that the ellipsoid centered at the origin that contains these k points and has the minimum value of the longest axis.

(b) (5 points) Prove that this problem can be formulated as the following SDP.

$$\min_{t\in\mathbb{R}, \mathbf{P}\in\mathbb{S}^n_{++}} \quad t \tag{4a}$$

$$\text{s.t. } \mathbf{P} \preceq t\mathbf{I}_n, \tag{4b}$$

$$\mathbf{P} \succeq \mathbf{v}_i\mathbf{v}_i^T, \ i = 1, \ldots, k. \tag{4c}$$

(c) (5 points) Derive the Lagrange dual function of problem (4).

(d) (5 points) Does strong duality hold for problem (4)? Justify your answer.

Problem 7. (20 points) Let $f : X \to \mathbb{R}$, where X is a Euclidean space (finite dimensional inner product space). The conjugate function of f is defined as

$$f^*(\mathbf{y}) = \sup_{\mathbf{x}\in X} \left\{\mathbf{y}^T\mathbf{x} - f(\mathbf{x})\right\}, \tag{5}$$

where $\mathbf{dom}\ f^* = \{\mathbf{y} \mid f^*(\mathbf{y}) < \infty\}$. Moreover, let us define an operator $\mathcal{M}_f$ as

$$\mathcal{M}_f(\mathbf{x}) = \arg\min_{\mathbf{u}\in X} \left\{f(\mathbf{u}) + \frac{1}{2}\|\mathbf{u} - \mathbf{x}\|^2\right\}, \quad \forall\, \mathbf{x} \in X. \tag{6}$$

(a) (5 points) Show that: $f(\mathbf{x}) + f^*(\mathbf{y}) \geq \mathbf{x}^T\mathbf{y}$ for all $\mathbf{x} \in X$ and $\mathbf{y} \in \mathbf{dom}\ f^*$.

(b) (8 points) Let $\alpha > 0$. Then show that:

$$\mathcal{M}_{(\alpha f)^*}(\mathbf{x}) = \alpha \mathcal{M}_{\frac{1}{\alpha} f^*}(\frac{1}{\alpha}\mathbf{x}).$$

(c) (7 points) Let $g(\mathbf{x}) \triangleq f(\mathbf{A}\mathbf{x} + \mathbf{b})$ where $\mathbf{A}$ is an invertible matrix. Find the conjugate of g based on the based on the conjugate of f (You have to simplify your answer as much as possible).