Convex Optimization for Communications and Signal Processing **Fall 2018**
Institute of Communications Engineering
Department of Electrical Engineering
National Tsing Hua University

Homework #2

Coverage: chapter 1–3

Due date: 23 November, 2018

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Notice:

1. *Please hand in the hardcopy of your answer sheets to the TAs by yourself before 23:59 of the due date. No late homework will be accepted.*
2. *Ansewrs to the problem set should be written on the A4 papers.*
3. *Write your name, student ID, email and department on the begining of your ansewr sheets.*
4. *You should have a clear idea to solve a problem. Please support your answers with clear, logical and solid reasoning or proofs.*
5. *Please do the homework independently by yourself. However, you may discuss with someone else but copyied homework is not allowed. This will show your respect toward the academic integrity.*
6. *Your legible handwriting is fine. However, you are very welcome to use text formatting packages (e.x., LaTeX) for writing your answers.*
7. *Please write your best solution for each problem. There will be bonus for the creative and beautiful solutions!*

Problem 1. (4 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a decreasing and convex function on its domain (a, b). Let g denote its inverse function. Is g a convex function? Justify your answer.

Problem 2. (6 points)

(a) (3 points) Let $f_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \ldots, n$, be quasi-convex functions, and $w_i \geq 0$ for $i = 1, \ldots, n$. Show that the function $f : \mathbb{R}^m \to \mathbb{R}$ defined as

$$f(\mathbf{x}) \triangleq \max\{w_1 f_1(\mathbf{x}), \ldots, w_n f_n(\mathbf{x})\}$$

is quasi-convex.

(b) (3 points) Let $g : \mathbb{R}^m \to \mathbb{R}$ be quasi-convex, and $h : \mathbb{R} \to \mathbb{R}$ is nondecreasing. Show that the function $f : \mathbb{R}^m \to \mathbb{R}$ defined as

$$f(\mathbf{x}) \triangleq h(g(\mathbf{x}))$$

is quasi-convex.

Problem 3. (15 points) Let $K \subseteq \mathbb{R}^m$ be a proper cone with associated generalized inequality $\preceq_K$, and let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$. For $\boldsymbol{\alpha} \in \mathbb{R}^m$, the $\boldsymbol{\alpha}$-sublevel set of $\boldsymbol{f}$ (with respect to $\preceq_K$) is defined as

$$C_{\boldsymbol{\alpha}} = \{\mathbf{x} \in \mathbb{R}^n | \boldsymbol{f}(\mathbf{x}) \preceq_K \boldsymbol{\alpha}\}.$$

The epigraph of $\boldsymbol{f}$, with respect to $\preceq_K$, is defined as the set

$$\mathbf{epi}_K\ \boldsymbol{f} = \{(\mathbf{x}, \mathbf{t}) \in \mathbb{R}^{n+m} | \boldsymbol{f}(\mathbf{x}) \preceq_K \mathbf{t}\}.$$

Show the following:

(a) (4 points) If $\boldsymbol{f}$ is K-convex, then its sublevel sets $C_{\boldsymbol{\alpha}}$ are convex for all $\boldsymbol{\alpha}$.

(b) (5 points) $\boldsymbol{f}$ is K-convex if and only if $\mathbf{epi}_K\ \boldsymbol{f}$ is a convex set.

(c) (6 points) $\mathbf{B}^T \mathbf{A}^{-1} \mathbf{B}$ is $\mathbb{S}^k_+$-convex on $\{(\mathbf{A}, \mathbf{B}) \in \mathbb{S}^k_{++} \times \mathbb{R}^{k \times (n-k)}\}$.

Problem 4. (10 points)

(a) (5 points) Show that $f(\mathbf{X}, t) = nt(\log t) - t \log\det \mathbf{X}$, with $\mathbf{dom}\ f = \mathbb{S}^n_{++} \times \mathbb{R}_{++}$, is convex in $(\mathbf{X}, t)$.

(b) (5 points) Use the result obtained in (a) to show that

$$g(\mathbf{X}) = n(\mathrm{Tr}(\mathbf{X})) \log(\mathrm{Tr}(\mathbf{X})) - (\mathrm{Tr}(\mathbf{X})) \log\det \mathbf{X}$$

is convex on $\mathbb{S}^n_{++}$.

Problem 5. (10 points) Show that the following functions are convex/concave.

(a) (5 points) $f(\mathbf{X}) = (\det(\mathbf{X}))^{1/m}$ with $\mathbf{dom}\ f = \mathbb{S}^m_{++}$.

(b) (5 points) $f(\mathbf{x}) = (\det(\mathbf{A}_0 + x_1 \mathbf{A}_1 + \cdots + x_n \mathbf{A}_n))^{1/m}$, on $\{\mathbf{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n |\ \mathbf{A}_0 + x_1 \mathbf{A}_1 + \cdots + x_n \mathbf{A}_n \succ 0\}$, where $\mathbf{A}_i \in \mathbb{S}^m$.

Problem 6. (25 points) Let $\mathbf{dom}\, f = \mathcal{B} \triangleq \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\|_1 \leq 1\}$. Using the definition of *convex envelope*, show that:

$$f(\mathbf{x}) = \|\mathbf{x}\|_0,\ \mathbf{x} \in \mathcal{B} \ \leftrightarrow\ g_f(\mathbf{x}) = \|\mathbf{x}\|_1,\ \mathbf{x} \in \mathcal{B} \tag{1}$$

where $g_f(\mathbf{x})$ is the convex envelope of f.

Problem 7. (10 points) Show that the following functions are convex/concave:

(a) (6 points) $f(\mathbf{X}) = \sup_{\mathbf{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \frac{\|\mathbf{X}\mathbf{v}\|_2}{\|\mathbf{v}\|_2}$ on $\mathbb{R}^{m \times n}$.

(b) (4 points) $f(\mathbf{x}, \mathbf{Y}) = \mathbf{x}^T \mathbf{Y}^{-1} \mathbf{x}$ on $\mathbb{R}^n \times \mathbb{S}^n_{++}$.

Problem 8. (20 points) Let

$$f(\mathbf{x}) = \mathrm{Tr}\left((\mathbf{P} + \mathbf{Diag}(\mathbf{x}))^{-1}\right), \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^n_{++}$, $\mathbf{P} \in \mathbb{S}^n_{++}$ and $\mathbf{Diag}(\mathbf{x})$ is an $n \times n$ diagonal matrix with the ith diagonal element equal to the ith element of $\mathbf{x}$. Is f a convex function? Justify your answer.