Convex Optimization for Communications and Signal Processing **Fall 2018**
Institute of Communications Engineering
Department of Electrical Engineering
National Tsing Hua University

Homework #1

Coverage: chapter 1–2

Due date: 26 October, 2018

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Notice:

1. *Please hand in the hardcopy of your answer sheets to the TAs by yourself before 23:59 of the due date. No late homework will be accepted.*
2. *Ansewrs to the problem set should be written on the A4 papers.*
3. *Write your name, student ID, email and department on the begining of your ansewr sheets.*
4. *You should have a clear idea to solve a problem. Please support your answers with clear, logical and solid reasoning or proofs.*
5. *Please do the homework independently by yourself. However, you may discuss with someone else but copyied homework is not allowed. This will show your respect toward the academic integrity.*
6. *Your legible handwriting is fine. However, you are very welcome to use text formatting packages (e.x., LaTeX) for writing your answers.*

Problem 1. (15 points) The Frobenius norm defined for $\mathbf{A} \in \mathbb{C}^{n\times n}$ by $\|\mathbf{A}\|_F = \left(\mathrm{Tr}(\mathbf{A}^H\mathbf{A})\right)^{1/2}$ where $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix. Show that

$$\|\mathbf{A}\|_F \le (\mathrm{rank}(\mathbf{A}))^{1/2}\|\mathbf{A}\|_2,$$

where $\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2=1}\|\mathbf{A}\mathbf{x}\|_2$ and $\mathbf{x}$ is an $n \times 1$ vector.

Problem 2. (5 points) Let $\mathcal{U} = \left\{\mathbf{x} \in \mathbb{R}^n_+ \mid \prod_{i=1}^n x_i \ge 1\right\}$. Is $\mathcal{U}$ a convex set? Justify your answer.

Problem 3. (7 points) Consider the sets $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C} \subseteq \mathbb{R}^n$, where $\mathcal{A}$ is a closed set. Assume that $\mathcal{H}$ is a supporting hyperplane of $\mathcal{A}$ and $\mathcal{C}$. Prove that: $(\mathbf{bd}\ \mathcal{A}) \cap \mathcal{H} \subseteq \mathbf{bd}\ \mathcal{B}$.

Problem 4. (8 points) Let K_i for $i = 1, \cdots, n$ be cones.

(a) (4 points) Is $\mathcal{K}_1 \triangleq \cap_{i=1}^n K_i$ a cone? Justify your answer.

(b) (4 points) Let $\mathcal{K}_2 \triangleq \{\mathbf{x} = \sum_{i=1}^n \mathbf{x}_i | \mathbf{x}_i \in K_i\}$. Is $\mathcal{K}_2$ a cone? Justify your answer.

Problem 5. (10 points) Let $\mathcal{F} = \{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1^2 \le x_2\}$. Is this a convex set? Justify your answer.

Problem 6. (15 points) Suppose $C \subset \mathbb{R}^n$ is a convex set and assume $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} \notin \mathbf{cl}\ C$. Prove that there exists a vector $\mathbf{a} \in \mathbb{R}^n$ such that $\langle \mathbf{a}, \mathbf{y}\rangle > \langle \mathbf{a}, \mathbf{x}\rangle$ for all $\mathbf{y} \in C$ (here: $\langle \mathbf{x}, \mathbf{y}\rangle \triangleq \mathbf{x}^T\mathbf{y}$).

Problem 7. (10 points) Show that the conic hull of any arbitrary set $C \subset \mathbb{R}^n$ can be represented as

$$\mathbf{conic}\ C = \{\theta\mathbf{v} \mid \mathbf{v} \in \mathbf{conv}\ C, \theta \in \mathbb{R}_+\},$$

where **conv** C is the convex hull of the C.

Problem 8. (5 points) Let $\mathcal{P} \subseteq \mathbb{R}^n$ be defined as

$$\mathcal{P} \triangleq \left\{\mathbf{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n x_i\mathbf{A}_i \preceq_K \mathbf{B}\right\},$$

where $\mathbf{A}_i \in \mathbb{R}^{m\times m}$, $\forall i \in \{1, \ldots, n\}$, $\mathbf{B} \in \mathbb{R}^{m\times m}$, and K is a proper cone. Is $\mathcal{P}$ a convex set?

Problem 9. (15 points) Let C be a set in $\mathbb{R}^n$. For any $\mathbf{x} \in \mathbf{conv}\ C$, prove that it can be represented as

$$\mathbf{x} = \sum_{i=1}^d \theta_i\mathbf{x}_i, \quad \theta_i \in [0,1],\ \sum_{i=1}^d \theta_i = 1,$$

where $\mathbf{x}_i \in C$ such that $d \le n + 1$.

Problem 10. (10 points) Let $K \subseteq \mathbb{R}^n$ be a proper cone and K^* is its dual. If $\boldsymbol{y} \in \mathbf{bd}\ K^*$ and $\boldsymbol{y} \ne \mathbf{0}$, show that there exists a $\boldsymbol{\lambda} \in K \setminus \{\mathbf{0}\}$ such that $\boldsymbol{\lambda}^T\boldsymbol{y} = 0$.