10810COM 524500

Convex Optimization for Communications and Signal Processing

Homework 1

Due: 01 Nov 2019 (12:05 PM)

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NOTE

- Every student must submit their hard-copy solution **by herself (himself) on the due date**. Late submission will be accepted (at ECE Room 706) but with point reduction (5% for each hour).
- As a NTHU student, strong academic ethics is assumed and punished otherwise by deducting the whole homework score.

Name: ____________________

id: ____________________

Total points: 100 10 Questions

Q1 . . . . (total: 10 Points)

(a) Consider the nonempty set $S \subset \mathbb{R}^n$. Prove that $(\theta_1+\theta_2)S = \theta_1 S + \theta_2 S$, $(\theta_1, \theta_2 \in \mathbb{R}_{++})$, when S is convex. (5 pt.)

(b) Consider the nonempty set $S \subset \mathbb{R}^n$, prove or disprove that $(\theta_1 + \theta_2)S = \theta_1 S + \theta_2 S$ when S is not convex. (5 pt.)

Q2 . . . . (total: 10 Points)

(a) Take nonempty bounded set $S \subset \mathbb{R}^n$. Prove that $\mathbf{cl}\,\mathbf{conv}\, S = \mathbf{conv}\,\mathbf{cl}\, S$. (4 pt.)

(b) Consider the two convex sets of S_1 and S_2. Prove (6 pt.)

$$\mathbf{relint}\,(S_1 \cap \mathbf{aff}\, S_2) \cap \mathbf{relint}\, S_2 \neq \emptyset \quad \Leftrightarrow \quad S_1 \cap \mathbf{relint}\, S_2 \neq \emptyset.$$

Q3 . . . . (total: 10 Points)

From basic set theory, a partition of the set A into two subsets is any two nonempty subsets A_1 and A_2, such that:

$$A_1 \cup A_2 = A \quad \& \quad A_1 \cap A_2 = \emptyset.$$

Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{N+2}\}$ be a set of distinct points ($\mathbf{a}_j \in \mathbb{R}^N$, $j = 1, 2, \ldots, N+2$, $N \in \mathbb{Z}_{++}$). Prove that for any A, there is always a partition of A into two subsets, such that: $\mathbf{conv}\, A_1 \cap \mathbf{conv}\, A_2 \neq \emptyset$.

Q4 . . . . (total: 10 Points)

Consider the set of points, $X \triangleq \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and point $\mathbf{q}$, all in $\mathbb{R}^d$. Let $\mathcal{H}(\mathbf{q} - \mathbf{x}_i, \mathbf{x}_i)$ be the hyperplane perpendicular to the line segment between $\mathbf{q}$ and $\mathbf{x}_i$ while goes through $\mathbf{x}_i$.

Define $H_{\mathbf{q}}(\mathbf{x}_i)$ as the halfspace that does not contain the point $\mathbf{q} \in \mathbb{R}^d$ and bounded by hyperplane $\mathcal{H}(\mathbf{q} - \mathbf{x}_i, \mathbf{x}_i)$. Show that

$$\bigcap_{i=1}^{n} H_{\mathbf{q}}(\mathbf{x}_i) = \emptyset \quad \Leftrightarrow \quad \mathbf{q} \in \mathbf{conv}\, X.$$

Q5 .... (total: 10 Points)

Consider $V = \{\mathbf{x} \,|\, \mathbf{A}\mathbf{x} \succeq \mathbf{0}\}$ which is a cone. Prove that $V^* = \{\mathbf{A}^T\mathbf{v} \,|\, \mathbf{v} \succeq \mathbf{0}\}$ which is the dual of V.

Hint

You might need to use the fact $(C \cap D)^* = C^* + D^*$ where C and D are cones and their duals are C^* and D^*, respectively.

Q6 .... (total: 10 Points)

(3 pt.) (a) Represent the set $\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_+ \,|\, x_1 x_2 \geq 1\}$, as the intersection of some family of halfspaces.

(3 pt.) (b) Suppose that C and D are disjoint subsets of $\mathbb{R}^n$. Show that the set,

$$A = \left\{(\mathbf{a}, b) \in \mathbb{R}^{n+1} \,|\, \mathbf{a}^T\mathbf{x} \leq b \,\forall \mathbf{x} \in C, \;\; \mathbf{a}^T\mathbf{x} \geq b \,\forall \mathbf{x} \in D\right\},$$

is a **convex cone**.

(4 pt.) (c) Consider two nonempty interior convex cones K_1 and K_2. Show that if $\mathbf{int}\, K_1 \cap \mathbf{int}\, K_2 = \emptyset$, then there is $\mathbf{y} \neq \mathbf{0}$ such that, $\mathbf{y} \in K_1^*$, $-\mathbf{y} \in K_2^*$.

Q7 .... (total: 10 Points)

Let two nonempty sets, K and C, be subsets of $\mathbb{R}^n$. At least one of them is a cone and $\mathbf{cl}\, K \cap \mathbf{cl}\, C \neq \emptyset$. Consider a hyperplane, $\mathcal{H}(\mathbf{a} \in \mathbb{R}^n, b)$, which separates K and C such that,

$$b = \sup\left\{\mathbf{a}^T\mathbf{x} \,|\, \mathbf{x} \in K\right\}.$$

Prove that the hyperplane passes through the origin.

Q8 .... (total: 10 Points)

Consider the two matrices $\mathbf{A}$ and $\mathbf{B}$ with the same number of rows. Let $\mathbf{conic}\, \mathbf{A}$ denote the conic hull of columns of $\mathbf{A}$. Prove there exists $\boldsymbol{P} \succeq \mathbf{0}$ such that $\mathbf{B} = \mathbf{A}\boldsymbol{P}$ *if and only if* $\mathbf{conic}\, \mathbf{B} \subseteq \mathbf{conic}\, \mathbf{A}$. Here "$\succeq$" stands for componentwise inequality.

Q9 .... (total: 10 Points)

Let $p, q \in \mathbb{Z}_{++}$. Denote $\mathbf{1}_p$ as all-one column vector of dimension p. Let,

$$L = \left\{(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^p \times \mathbb{R}^q \,\middle|\, \mathbf{x} \succeq_{\mathbb{R}^p_+} \|\mathbf{u}\|_2 \mathbf{1}_p\right\} \tag{9}$$

$$Z = \left\{(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^p \times \mathbb{R}^q \,\middle|\, \mathbf{x}^T\mathbf{1}_p \geq \|\mathbf{u}\|_2,\; \mathbf{x} \succeq_{\mathbb{R}^p_+} \mathbf{0}\right\} \tag{10}$$

Prove that $Z = L^*$ (*aka.* Z is the dual cone of L).

Q10 . . . . (total: 10 Points)

Consider the entrywise matrix product operator, $\odot$, defined as $[\mathbf{A} \odot \mathbf{B}]_{ij} \triangleq [\mathbf{A}]_{ij}[\mathbf{B}]_{ij}$. Define the set $S \triangleq \left\{ \mathbf{A} \odot \mathbf{I}_n - \mathbf{A} \odot \mathbf{A} \;\middle|\; \mathbf{A}^2 = \mathbf{A},\ \mathbf{A} \in \mathbb{S}^n \right\}$.

(a) Prove that S is a subset of $\mathbb{S}^n_+$. (6 pt.)

(b) Show whether the set S is a convex cone or not. (4 pt.)

References

[1] C.-Y. Chi, W.-C. Li, and C.-H. Lin, *Convex optimization for signal processing and communications: from fundamentals to applications*. CRC Press, 2019, (draft version: 13 June 2019).