COM 5245
Convex Optimization for Signal Processing and Communications
Fall 2017
Homework #3 (Due date: December 27, 2017)

Notations:

$\mathbf{x} = [\ x_1, \ldots, x_n\ ]^T$	n-dimensional column vector
$\mathbf{X} = [\ \mathbf{x}_1, \ldots, \mathbf{x}_m\ ]$	$n \times m$ matrix
dom $\boldsymbol{f}$	Domain of a function $\boldsymbol{f}$

1. (10 points) Consider the following convex problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} \ \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$$
$$\text{s.t. } \mathbf{G}\mathbf{x} = \mathbf{h},$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $\text{rank}(\mathbf{A}) = n$, and $\mathbf{G} \in \mathbb{R}^{p \times n}$ with $\text{rank}(\mathbf{G}) = p$.

   (a) (5 points) Find the KKT conditions.

   (b) (5 points) Find the primal and dual optimal solution $\mathbf{x}^\star$, $\boldsymbol{\nu}^\star$ by solving the KKT conditions.

2. (15 points) Consider the convex piecewise-linear minimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} \ \max_{i=1,\ldots,m} (\mathbf{a}_i^T \mathbf{x} + b_i). \tag{1}$$

   (a) (5 points) Find the epigraph reformulation of this problem.

   (b) (10 points) Derive the Lagrange dual problem of the above epigraph reformulation.

3. (15 points) Derive the dual problem for the optimization problem defined as

$$\min_{\mathbf{x}} \ -\sum_{i=1}^{m} \log(b_i - \mathbf{a}_i^T \mathbf{x})$$

with domain $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_i^T \mathbf{x} < b_i,\ i = 1, \ldots, m\}$.

4. (10 points) Express the dual problem of

$$\min \ \mathbf{c}^T \mathbf{x}$$
$$\text{s.t. } f(\mathbf{x}) \le 0,$$

with $\mathbf{c} \neq \mathbf{0}$, in terms of the conjugate function f^*.

5. (15 points) Consider the optimization problem

$$\min_{x \in \mathbb{R}, y \in \mathbb{R}} \ e^{-x}$$
$$\text{s.t. } x^2/y \le 0$$

with problem domain $\mathcal{D} = \{(x, y) \mid\ y > 0\}$.

(a) (5 points) Verify that this is a convex optimization problem, and find the optimal solution $(x^\star, y^\star)$ and the optimal value $p^\star$.

(b) (5 points) Derive the Lagrange dual problem, and find the optimal solution $\lambda^\star$ and the optimal value $d^\star$ of the dual problem. What is the duality gap?

(c) (5 points) Does Slater's condition hold for this problem? Why?

6. (20 points) We denote by $f(\mathbf{A})$ the sum of the largest r eigenvalues of a symmetric matrix $\mathbf{A} \in \mathbb{S}^n$ (with $1 \leq r \leq n$), i.e.,

$$f(\mathbf{A}) = \sum_{k=1}^{r} \lambda_k(\mathbf{A}),$$

where $\lambda_1(\mathbf{A}), \ldots, \lambda_n(\mathbf{A})$ are the eigenvalues of $\mathbf{A}$ sorted in decreasing order.

(a) (10 points) Show that the optimal value of the SDP

$$\begin{aligned} \max\ & \operatorname{tr}(\mathbf{AX}) \\ \text{s.t.}\ & \operatorname{tr}(\mathbf{X}) = r, \\ & \mathbf{0} \preceq \mathbf{X} \preceq \mathbf{I}, \end{aligned}$$

with variable $\mathbf{X} \in \mathbb{S}^n$, is equal to $f(\mathbf{A})$.
(*Hint:* Show that strong duality holds for the above SDP, and obtain the optimal value by solving the dual problem.)

(b) (5 points) Show that f is a convex function.

(c) (5 points) Let $\mathbf{A}(\mathbf{x}) = \mathbf{A}_0 + x_1\mathbf{A}_1 + \cdots + x_m\mathbf{A}_m$, with $\mathbf{A}_k \in \mathbb{S}^n$. Reformulate the optimization problem

$$\min\ f(\mathbf{A}(\mathbf{x})),$$

with variable $\mathbf{x} \in \mathbb{R}^m$, as an SDP.

7. (15 points) The relative entropy between two vectors $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^n_{++}$ is defined as

$$\sum_{k=1}^{n} x_k \log(x_k/y_k).$$

This is a convex function, jointly in $\mathbf{x}$ and $\mathbf{y}$. In the following problem we calculate the vector $\mathbf{x}$ that minimizes the relative entropy with a given vector $\mathbf{y}$, subject to equality constraints on $\mathbf{x}$:

$$\begin{aligned} \min\ & \sum_{k=1}^{n} x_k \log(x_k/y_k) \\ \text{s.t.}\ & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{1}_n^T\mathbf{x} = 1. \end{aligned}$$

The optimization variable is $\mathbf{x} \in \mathbb{R}^n$. The domain of the objective function is $\mathbb{R}^n_{++}$. The parameters $\mathbf{y} \in \mathbb{R}^n_{++}$, $\mathbf{A} \in \mathbb{R}^{m \times n}$, and $\mathbf{b} \in \mathbb{R}^m$ are given. Derive the Lagrange dual of this problem and simplify it to get

$$\max_{\mathbf{z} \in \mathbb{R}^m}\ \mathbf{b}^T\mathbf{z} - \log\left(\sum_{k=1}^{n} y_k e^{\mathbf{a}_k^T\mathbf{z}}\right)$$

($\mathbf{a}_k$ is the kth column of $\mathbf{A}$).