COM 5245
Optimization for Communications - Fall 2017
Homework 2
(Due date: November 03, 2017)

Notations:

$\mathbf{x} = [\ x_1, \ldots, x_n\ ]^T$	n-dimensional column vector
$\mathbf{X} = [\ \mathbf{x}_1, \ldots, \mathbf{x}_m\ ]$	$n \times m$ matrix
$\mathbf{dom}\ \boldsymbol{f}$	Domain of a function $\boldsymbol{f}$

A function f is called log-convex/ log-concave if

(1) $f(\mathbf{x}) > 0,\ \forall \mathbf{x} \in \mathbf{dom}\ f$

(2) $\ln(f(\mathbf{x}))$ is convex/concave.

$\|\mathbf{x}\|_p^p = \sum_{i=1}^n |x_i|^p$ the pth power of the l_p-norm of a vector $\mathbf{x} \in \mathbb{R}^n$

1. (30 points)
   (a) (10 points) Show that $\|\mathbf{x}\|_p^p$ is a convex function on $\mathbb{R}^n$ for $p > 1$.
   (b) (5 points) Show that
   $$f(\mathbf{x}, t) \triangleq \frac{\|\mathbf{x}\|_p^p}{t^{p-1}}$$
   is convex on $\{(\mathbf{x}, t) |\ \mathbf{x} \in \mathbb{R}^n,\ t > 0\}$.
   (c) (15 points) Show that $f(\mathbf{X}) = \mathrm{Tr}(\mathbf{X}^{-1})$ is convex on $\mathbf{dom}\ f = \mathbb{S}_{++}^n$.

2. (15 points) Verify quasiconvexity of the function $f(\mathbf{x}) = -x_1 x_2$, with $\mathbf{dom}\ f = \mathbb{R}_{++}^2$ by using
   (a) (10 points) first-order condition for quasiconvexity.
   (b) (5 points) second-order condition for quasiconvexity.

3. (20 points) Show that
   $$f(\mathbf{x}) = \max_{k=1,\ldots,m} \{\sum_{i=1}^P c_{ki} a_{1ki}^{x_1} a_{2ki}^{x_2} \ldots a_{nki}^{x_n}\}$$
   is a convex function in $\mathbf{x}$, where $\mathbf{x} = [x_1, \ldots, x_n]^T,\ c_{ki}, a_{jki} \in \mathbb{R}_{++},\ \forall j = 1, \ldots, n,\ i = 1, \ldots, P,\ k = 1, \ldots, m$.

4. (10 points) Let $K \subseteq \mathbb{R}^m$ be a proper cone, and K^* denotes its dual cone. Show that a function $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ is K-convex if and only if for every $\mathbf{w} \succeq_{K^*} \mathbf{0}$, the function $\mathbf{w}^T \boldsymbol{f}$ is convex.

5. (10 points)
   (a) (5 points) Suppose that $f_i : \mathbb{R}^m \to \mathbb{R},\ i = 1, \ldots, n$, are quasi-convex functions, and $w_i \geq 0$ for $i = 1, \ldots, n$. Show that the function $f : \mathbb{R}^m \to \mathbb{R}$ defined as
   $$f(\mathbf{x}) \triangleq \max\{w_1 f_1(\mathbf{x}), \ldots, w_n f_n(\mathbf{x})\}$$
   is quasi-convex.

(b) (5 points) Suppose that $g : \mathbb{R}^m \to \mathbb{R}$ is quasi-convex, and $h : \mathbb{R} \to \mathbb{R}$ is nondecreasing. Show that the function $f : \mathbb{R}^m \to \mathbb{R}$ defined as

$$f(\mathbf{x}) \triangleq h(g(\mathbf{x}))$$

is quasi-convex.

6. (15 points) Let $K \subseteq \mathbb{R}^m$ be a proper cone with $\preceq_K$ being the associated generalized inequality, and let $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$. For $\boldsymbol{\alpha} \in \mathbb{R}^m$, the $\boldsymbol{\alpha}$-sublevel set of $\boldsymbol{f}$ (with respect to $\preceq_K$) is defined as

$$C_{\boldsymbol{\alpha}} = \{\mathbf{x} \in \mathbb{R}^n | \boldsymbol{f}(\mathbf{x}) \preceq_K \boldsymbol{\alpha}\}.$$

The epigraph of $\boldsymbol{f}$, with respect to $\preceq_K$, is defined as the set

$$\mathbf{epi}_K\ \boldsymbol{f} = \{(\mathbf{x}, \mathbf{t}) \in \mathbb{R}^{n+m} | \boldsymbol{f}(\mathbf{x}) \preceq_K \mathbf{t}\}.$$

Show the following:

(a) (5 points) If $\boldsymbol{f}$ is K-convex, then its sublevel sets $C_{\boldsymbol{\alpha}}$ are convex for all $\boldsymbol{\alpha}$.

(b) (10 points) $\boldsymbol{f}$ is K-convex if and only if $\mathbf{epi}_K\ \boldsymbol{f}$ is a convex set.