COM 5245
Optimization for Communications - Fall 2017
Homework 1
(Due date: October 18, 2017)

Notations:

$\mathbf{x} = [\ x_1, \ldots, x_n\ ]^T$	n-dimensional column vector
$\mathbf{X} = [\ \mathbf{x}_1, \ldots, \mathbf{x}_m\ ]$	$n \times m$ matrix
dom $\boldsymbol{f}$	Domain of a function $\boldsymbol{f}$

1. (20 points) Which of the following sets are convex? (You need to justify your answer with strong and clear reasoning.)

   (a) (5 points) The set of points closer to a given point than a given set, i.e.,

   $$\{\mathbf{x} |\ \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2\ \forall\ \mathbf{y} \in S\},$$

   where $S \subseteq \mathbb{R}^n$.

   (b) (5 points) The set of points closer to one set than another, i.e.,

   $$\{\mathbf{x} |\ \mathsf{dist}(\mathbf{x}, S) \leq \mathsf{dist}(\mathbf{x}, T)\},$$

   where S, $T \subseteq \mathbb{R}^n$, and

   $$\mathsf{dist}(\mathbf{x}, S) \triangleq \inf_{\mathbf{z}} \{\|\mathbf{x} - \mathbf{z}\|_2 |\ \mathbf{z} \in S\}.$$

   (c) (5 points) The set

   $$\{\mathbf{x} |\ (\mathbf{x} + S_2) \subseteq S_1\} = \{\mathbf{x} |\ \mathbf{x} + \mathbf{y} \in S_1,\ \forall\ \mathbf{y} \in S_2\},$$

   where S_1, $S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

   (d) (5 points) The set of points whose distance to a point $\mathbf{a}$ does not exceed a fixed fraction $0 \leq \theta \leq 1$ of the distance to another point $\mathbf{b}$, i.e., the set

   $$\{\mathbf{x} |\ \|\mathbf{x} - \mathbf{a}\|_2 \leq \theta \|\mathbf{x} - \mathbf{b}\|_2\}.$$

2. (10 points) Which of the following sets S are polyhedra? If possible, express S in the form $S = \{\mathbf{x} | \mathbf{A}\mathbf{x} \preceq \mathbf{b}, \mathbf{F}\mathbf{x} = \mathbf{g}\}$.

   (a) (5 points) $S = \{\mathbf{x} \in \mathbb{R}^n |\ \mathbf{x} \succeq \mathbf{0},\ \mathbf{1}^T\mathbf{x} = 1,\ \sum_{i=1}^n x_i a_i = b_1,\ \sum_{i=1}^n x_i a_i^2 = b_2\}$, where $a_1, \ldots, a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.

   (b) (5 points) $S = \{\mathbf{x} \in \mathbb{R}^n |\ \mathbf{x} \succeq \mathbf{0},\ \mathbf{x}^T\mathbf{y} \leq 1$ for all $\mathbf{y}$ with $\|\mathbf{y}\|_2 = 1\}$.

3. (10 points) The dual norm of $\|\cdot\|$ on $\mathbb{R}^n$ is defined as

   $$\|\mathbf{x}\|_* = \sup\ \{\mathbf{x}^T\mathbf{y}\ |\ \|\mathbf{y}\| \leq 1,\ y \in \mathbb{R}^n\}.$$

   Prove that $\|\cdot\|_*$ is a valid norm. (Hint: try to show the essential properties of a norm including positive definiteness, positive homogeneity and triangle inequality.)

4. (5 points) Let K be a nonempty cone in $\mathbb{R}^n$. Suppose that f_K is the image of K under a linear mapping. Show that f_K is a cone.

5. (15 points) Suppose that C and D are closed convex cones in $\mathbb{R}^n$ and C^* and D^* are the associated dual cones. Show that

$$(C \cap D)^* = C^* + D^*.$$

6. (20 points) Let K be a nonempty cone in $\mathbb{R}^n$. Show that: $K^* = (\mathbf{cl}\ \mathbf{conv} K)^*$.

7. (10 points) Let $C \subseteq \mathbb{R}^n$ be a closed convex set, and suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_K$ are on the boundary of C. Suppose that for each i, $\mathbf{a}_i^T(\mathbf{x} - \mathbf{x}_i) = 0$ defines a supporting hyperplane for C at $\mathbf{x}_i$, i.e., $C \subseteq \{\mathbf{x} \mid \mathbf{a}^T(\mathbf{x} - \mathbf{x}_i) \leq 0\}$. Consider the two polyhedra

$$P_{\text{inner}} = \mathbf{conv}\{\mathbf{x}_1, \ldots, \mathbf{x}_K\}, \qquad P_{\text{outer}} = \{\mathbf{x} \mid \mathbf{a}^T(\mathbf{x} - \mathbf{x}_i) \leq 0,\ i = 1, \ldots, K\}.$$

Show that $P_{\text{inner}} \subseteq C \subseteq P_{\text{outer}}$. Draw a picture illustrating this.

8. (10 points) Let $\mathcal{S} = \{\mathbf{s}_1, ..., \mathbf{s}_{m+1}\} \subset \mathbb{R}^n,\ n \in \mathbb{Z}_+$ be an *affinly independent* set. The *m-simplex* $\Delta_m = \Delta(\mathbf{s}_1, ..., \mathbf{s}_{m+1})$ with vertices from $\mathcal{S}$ is defined as

$$\Delta_m = \{\sum_{i=1}^{m+1} \theta_i \mathbf{s}_i \mid \mathbf{s}_i \in \mathcal{S}, \theta_i \in \mathbb{R}_+, \sum_{i=1}^{m+1} \theta_i = 1\}.$$

Now, considering this definition,

(a) (5 points) Show that any $\Delta_m \subset \mathbb{R}^n$ is a convex set of dimension m;

(b) (5 points) Show that aff Δ_m = aff $\mathcal{S}$.