National Tsing Hua University
10810COM 524500
Convex Optimization for Communications and Signal Processing

Homework 4

Due: 03 Jan 2020 (12:05 PM)

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Total points: 100 — 6 Questions

Q1 .... (total: 15 Points)

Consider the optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R},y\in\mathbb{R}} \quad & e^{-x} \\ \text{s.t.} \quad & x^2/y \leq 0 \end{aligned}$$

with problem domain $\mathcal{D} = \{(x,y) |\ y > 0\}$.

(a) Verify that this is a convex optimization problem, and find the optimal solution $(x^\star, y^\star)$ and the optimal value $p^\star$. (5 pt.)

(b) Derive the Lagrange dual problem, and find the optimal solution $\lambda^\star$ and the optimal value $d^\star$ of the dual problem. What is the duality gap? (5 pt.)

(c) Discuss whether the Slater's condition does hold for this problem or not. (5 pt.)

Q2 .... (total: 20 Points)

Consider the following convex optimization problem

$$\begin{aligned} \min \quad & \log\det\left(\begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 \\ \mathbf{X}_2^T & \mathbf{X}_3 \end{bmatrix}^{-1}\right) \\ \text{s.t.} \quad & \mathrm{Tr}(\mathbf{X}_1) = \alpha, \\ & \mathrm{Tr}(\mathbf{X}_2) = \beta, \\ & \mathrm{Tr}(\mathbf{X}_3) = \gamma, \end{aligned}$$

where $\mathbf{X}_1 \in \mathbb{S}^n$, $\mathbf{X}_2 \in \mathbb{R}^{n\times n}$, $\mathbf{X}_3 \in \mathbb{S}^n$ are the variables. The domain of the objective function is $\mathbb{S}^{2n}_{++}$. Assume that $\alpha > 0$, and $\alpha\gamma > \beta^2$.

(a) Reformulate the problem by introducing new variable, $\mathbf{X} \triangleq \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 \\ \mathbf{X}_2^T & \mathbf{X}_3 \end{bmatrix}$. Then, derive the KKT conditions of the reformulated problem. (10 pt.)

(b) Solve the KKT conditions to obtain the optimal solution. (10 pt.)

Hint

Suppose a block matrix of the form

$$\mathbf{X} = \begin{bmatrix} a\mathbf{I}_n & b\mathbf{I}_n \\ c\mathbf{I}_n & d\mathbf{I}_n \end{bmatrix},$$

where $a, b, c, d \in \mathbb{R}$. The inverse of $\mathbf{X}$ can be obtained as

$$\mathbf{X}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d\mathbf{I}_n & -b\mathbf{I}_n \\ -c\mathbf{I}_n & a\mathbf{I}_n \end{bmatrix}.$$

Q3 .... (total: 15 Points)

Consider the following problem:

$$\min_{\mathbf{x} \in \mathbb{R}_+^n} \alpha \mathbf{1}_n^T \mathbf{x} + \frac{\eta}{2} \|\mathbf{s} - \mathbf{x}\|_2^2, \tag{2}$$

where α, η are nonnegative parameters and $\mathbf{s} \in \mathbb{R}_+^n$ is a given vector.

(10 pt.) (a) What are the KKT conditions of problem (2)?

(5 pt.) (b) Find the optimal $(\mathbf{x}^\star, \boldsymbol{\lambda}^\star)$ pair.

Q4 .... (total: 20 Points)

Consider the following optimization problem

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^n, s \in \mathbb{R}} \quad & s \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} - \mathbf{b} - s\mathbf{1}_m \preceq \mathbf{0}, \end{aligned} \tag{P4}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{1}_m$ denotes the m-dimensional all-one vector.

(5 pt.) (a) Derive the Lagrange dual problem.

(5 pt.) (b) Does the Slater's condition hold true? Prove your answer.

(10 pt.) (c) Define

$$\begin{aligned} \mathcal{A} &\triangleq \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \prec \mathbf{b}\}, \\ \mathcal{B} &\triangleq \{\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \succeq \mathbf{0}, \quad \boldsymbol{\lambda} \neq \mathbf{0}, \quad \mathbf{A}^T\boldsymbol{\lambda} = \mathbf{0}, \quad \mathbf{b}^T\boldsymbol{\lambda} \leq \mathbf{0}\}. \end{aligned}$$

Use the results in Part (a) and Part (b) to prove that $\mathcal{A}$ is an empty set if, and only if, $\mathcal{B}$ is a nonempty set.

Q5 .... (total: 20 Points)

Consider $f_0(\mathbf{X}) = \log\det \mathbf{X}^{-1}$ with $\mathbf{dom}\, f_0 = \mathbb{S}_{++}^n$.

(10 pt.) (a) Calculate the conjugate function of f_0.

Hint

Note that $f_0^*(\mathbf{Y}) = \sup_{\mathbf{X} \in \mathbf{dom}\, f} \{\mathrm{Tr}(\mathbf{Y}\mathbf{X}) - \log\det \mathbf{X}^{-1}\}$.

Question 5 may continues on the next page...

(b) Consider the problem, (10 pt.)

$$
\begin{aligned}
p^\star = \min_{\mathbf{X}} \quad & f_0(\mathbf{X}) \\
s.t. \quad & \mathbf{a}_i^T \mathbf{X} \mathbf{a}_i \le 1, \quad i = 1, \ldots, m.
\end{aligned}
$$

By calculating the Lagrange dual function, find a lower bound for the optimal value of the problem, $p^\star$.

Q6 .... (total: 10 Points)

A given undirected graph ($\mathcal{G}$) can be represented by nonnegative symmetric matrix. Consider symmetric matrix $\mathbf{W} \in \mathbb{S}^n$ with nonnegative elements where the diagonal elements are zero. The Laplacian of the graph is defined as $\mathbf{L}(\mathbf{W}) \triangleq \mathbf{Diag}\left(\mathbf{W}\mathbf{1}\right) - \mathbf{W}$. Consider the optimization problem,

$$
\begin{aligned}
\min_{\mathbf{x} \in \mathbb{R}^n} \quad & nt \\
s.t. \quad & \mathbf{L}(\mathbf{W}) + \mathbf{Diag}\left(\mathbf{x}\right) \preceq t\mathbf{I} \\
& \mathbf{1}^T \mathbf{x} = 0,
\end{aligned}
\tag{6}
$$

where $\mathbf{W} \in \mathbb{S}^n$ is a given matrix. Show that the dual problem of (6) is,

$$
\begin{aligned}
\max \quad & \operatorname{Tr}\left(\mathbf{L}(\mathbf{W})\mathbf{Z}\right) \\
s.t. \quad & \mathbf{vecdiag}\left(\mathbf{Z}\right) = \mathbf{1} \\
& \mathbf{Z} \succeq \mathbf{0}.
\end{aligned}
\tag{7}
$$

References

[1] C.-Y. Chi, W.-C. Li, and C.-H. Lin, *Convex optimization for signal processing and communications: from fundamentals to applications*. CRC Press, 2019, (draft version: 13 June 2019).