National Tsing Hua University
10810COM 524500
Convex Optimization for Communications and Signal Processing

Homework 3

Due: 25 Dec 2019 (12:05 PM)

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Total points: 100 | 9 Questions

Q1 .... (total: 25 Points)

Prove the following statements on Frobenius norm of a matrix $\|\mathbf{A}\|_F$ ([1, Eq. (1.6)]):

(a) $K = \mathbb{S}^n_+$, $f(\mathbf{A}) = \|\mathbf{A}\|_{\rm F}$ is K-nondecreasing. (5 pt.)

(b) $K = \mathbb{S}^n_{++}$, $f(\mathbf{A}) = \mathbf{A}^{-1}$ is K-convex. (5 pt.)

(c) $K = \mathbb{S}^n_{++}$, $f(\mathbf{A}) = \|\mathbf{A}^{-1}\|_F$ is convex. (5 pt.)

> Note
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> Note that part (b) has been solved in [1, **Example 3.13, 3.15**]. So, your proof for part (b) must be different from these two examples.

Q2 .... (total: 10 Points)

Consider $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{b}, \boldsymbol{\lambda} \in \mathbb{R}^m$. Define $\mathcal{A} \triangleq \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \prec \mathbf{b}\}$ and

$$\mathcal{B} \triangleq \{\boldsymbol{\lambda} \mid \boldsymbol{\lambda} \succeq \mathbf{0}, \quad \boldsymbol{\lambda} \neq 0, \quad \mathbf{A}^T\boldsymbol{\lambda} = \mathbf{0}, \quad \mathbf{b}^T\boldsymbol{\lambda} \leq \mathbf{0}\}.$$

(a) Prove that $\mathcal{C} = \{\mathbf{b} - \mathbf{A}\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$ is a convex set. (3 pt.)

(b) Apply the *separating hyperplane theorem* to prove that $\mathcal{A}$ is an empty set *if and only if*, $\mathcal{B}$ is a nonempty set. (7 pt.)

Q3 .... (total: 10 Points)

Let $K \subseteq \mathbb{R}^m$ be a proper cone, and K^* denotes its dual cone. Show that a function $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$ is K-convex *if and only if* for every $\mathbf{w} \succeq_{K^*} \mathbf{0}$, the function $\mathbf{w}^T\boldsymbol{f}$ is convex.

Q4 .... (total: 10 Points)

(a) Consider $f : \mathbb{R}^n \to \mathbb{R}$ as a convex function and denote $\mathbf{x}^\star$ as its global minimum. For all $\mathbf{y} \in \mathbb{R}^n$, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(\alpha) \triangleq f(\mathbf{x}^\star + \alpha\mathbf{y})$ is defined. Prove that $\mathbf{x}^\star$ is the global minimum of f *if and only if* $\forall \mathbf{y} \in \mathbb{R}^n$, $\alpha^\star = 0$ is the global minimum of the function $g(\alpha)$. (3 pt.)

(b) Let's consider a case where the function f is <u>nonconvex</u>. Denote $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{x}^\star = (0, 0)$ and $p, q \in \mathbb{R}_{++}$, $p < q$. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = (x_2 - px_1^2)(x_2 - qx_1^2)$. Show that if $f(y, my^2) < 0$ for $y \neq 0$ and m satisfying $p < m < q$, then $\mathbf{x}^\star$ is not a local minimum of f even though it is a local minimum along every line passing through $\mathbf{x}^\star$. (7 pt.)

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Q5 .... (total: 10 Points)

Consider the following optimization problem,

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \ & \sup_{\|\mathbf{c}\|_2\le 1} \mathbf{c}^T\mathbf{F}(\mathbf{x})^{-1}\mathbf{c} \\ \text{s.t.} \ & \mathbf{F}(\mathbf{x}) \succ \mathbf{0}, \end{aligned}$$

where

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_0 + x_1\mathbf{F}_1 + \cdots + x_n\mathbf{F}_n$$

and each $\mathbf{F}_i \in \mathbb{S}^m$. Reformulate it as the following,

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}^n, t\in\mathbb{R}} \ & t \\ \text{s.t.} \ & \mathbf{F}(\mathbf{x}) - t\mathbf{I} \succeq \mathbf{0}, \\ & \mathbf{F}(\mathbf{x}) \succ \mathbf{0}, \end{aligned}$$

Hint

You may take a look at [1, *(Maximum eigenvalue minimization)* Example 8.1].

Q6 .... (total: 5 Points)

Consider the complex least ℓ_2-norm problem,

$$\begin{aligned} \min_{\mathbf{x}} \ & \|\mathbf{x}\|_2 \\ \text{s.t.} \ & \mathbf{A}\mathbf{x} = \mathbf{b}, \end{aligned}$$

where $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{b} \in \mathbb{C}^m$, and the variable is $\mathbf{x} \in \mathbb{C}^n$. Here $\|\cdot\|_2$ denotes the ℓ_2-norm on $\mathbb{C}^n$, defined as

$$\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}.$$

Assume that $\mathbf{A}$ is full rank, and $m < n$. Formulate the complex least ℓ_2-norm problem as a least ℓ_2-norm problem with real problem data and variable.

Q7 .... (total: 10 Points)

Reformulate the following optimization problem,

$$\begin{aligned} \max_{\mathbf{p},\mathbf{r}\in\mathbb{R}^K} \ & \left(\prod_{k=1}^{K} r_k\right)^{1/K} \\ \text{s.t.} \ & r_k \le \ln\left(1 + \frac{p_k}{\sigma_k^2 + \sum_{j\ne k}^{K} \alpha_{k,j}p_j}\right), \ k = 1,\dots,K, \\ & 0 \le p_k \le P_k, \ k = 1,\dots,K, \end{aligned}$$

where all σ_k^2, $\alpha_{k,j}$, and P_k are given positive real numbers, into a convex problem.

Hint

The following functions are convex:

$$f(\mathbf{x}) = \ln\left(\sum_{k=1}^{K} a_k \exp(x_k)\right), \quad \textbf{dom } f = \mathbb{R}^K, \quad \text{where } a_k \geq 0 \text{ for } k = 1, \ldots, K,$$

$$g(t) = \ln(\exp(\exp(t)) - 1), \quad \textbf{dom } g = \mathbb{R}.$$

Q8 .... (total: 10 Points)

The function $f : \mathbb{R}^n \to \mathbb{R}$ is a quasiconvex function. Consider the convex set $X \subseteq \mathbb{R}^n$ and denote $p^\star = \inf_{\mathbf{x} \in X} f(\mathbf{x})$. Assume that f is not constant on any line segment of X. Prove that every local minimum of f over the set X, is also the global minimum.

Q9 .... (total: 10 Points)

Consider the optimization problem (P1),

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^n} \quad & \frac{\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_1}{\mathbf{c}^T\mathbf{x} + d} \\ s.t. \quad & \|\mathbf{x}\|_\infty \leq 1, \end{aligned} \tag{P1}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$, and $d \in \mathbb{R}$. Suppose $\mathbf{c}^T\mathbf{x} + d > 0$ for all feasible $\mathbf{x}$.

(a) Prove (P1) is a quasiconvex optimization problem. (4 pt.)

(b) Show that (P1) is equivalent to (P2), (6 pt.)

$$\begin{aligned} \min_{\mathbf{y} \in \mathbb{R}^n} \quad & \|\mathbf{A}\mathbf{y} - \mathbf{b}t\|_1 \\ s.t. \quad & \|\mathbf{y}\|_\infty \leq t \\ & \mathbf{c}^T\mathbf{y} + dt = 1, \end{aligned} \tag{P2}$$

where $t \in \mathbb{R}$.

References

[1] C.-Y. Chi, W.-C. Li, and C.-H. Lin, *Convex optimization for signal processing and communications: from fundamentals to applications*. CRC Press, 2019, (draft version: 13 June 2019).