10810COM 524500
Convex Optimization for Communications and Signal Processing

Homework 2

Due: 29 Nov 2019 (12:05 PM)

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NOTE

- Every student must submit their hard-copy solution **by herself (himself) on the due date**. Late submission will be accepted (at ECE Room 706) but with point reduction (5% for each hour).
- As a NTHU student, strong academic ethics is assumed and punished otherwise by deducting the whole homework score.

Name: ____________________

id: ____________________

Total points: 100 | 9 Questions

Q1 .... (total: 16 Points)

Consider $S \subseteq \mathbb{R}^n$ as a closed convex set. We know that for any $\mathbf{x} \in \mathbb{R}^n$, there always exists a nearest point from S. We denote this nearest point by $\mathbf{s}_0$.

(a) Prove $\mathbf{s}_0$ satisfies the following inequality for any $\mathbf{y} \in S$: (8 pt.)

$$(\mathbf{x} - \mathbf{s}_0)^T(\mathbf{y} - \mathbf{s}_0) \leq 0, \quad (1)$$

(b) Using the result from part (a), prove that for every $\mathbf{x} \in \mathbb{R}^n$ the nearest point $\mathbf{s}_0 \in S$ is unique. (8 pt.)

Q2 .... (total: 10 Points)

Show that $f(\mathbf{X}) = \mathrm{Tr}(\mathbf{X}^{-1})$ is convex on $\textbf{dom}\, f = \mathbb{S}^n_{++}$.

Q3 .... (total: 10 Points)

Consider $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$. Show that the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$f(\mathbf{x}) = \begin{cases} -(x_1 x_2 \ldots x_n)^{\frac{1}{n}}, & \text{if } x_i > 0,\ i = 1, 2, \ldots, n \\ \infty, & \text{otherwise} \end{cases},$$

is convex.

Q4 .... (total: 10 Points)

Consider $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ as a convex function for $i = 1, \ldots, m$. Let $\boldsymbol{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be K-convex ($K = \mathbb{R}^m_+$) where $\boldsymbol{f} = (f_1, \ldots, f_m)$ and $g : D \subseteq \mathbb{R}^m \rightarrow \mathbb{R}$ be a monotonically nondecreasing convex function over the convex set $D \supseteq \{\boldsymbol{f}(\mathbf{x}) \,|\, \mathbf{x} \in \mathbb{R}^n\}$. Show that composite function h defined as $h(\mathbf{x}) = g(\boldsymbol{f}(\mathbf{x}))$ is convex.

Q5 .... (total: 10 Points)

Verify quasiconvexity of the function $f(\mathbf{x}) = -x_1 x_2$, with $\textbf{dom}\ f = \mathbb{R}^2_{++}$ by using

(5 pt.) (a) first-order condition for quasiconvexity.

(5 pt.) (b) second-order condition for quasiconvexity.

Q6 .... (total: 10 Points)

Prove that for any arbitrary set $C \subset \mathbb{R}^n$ ([1, Eq. (2.30)]),

$$\begin{aligned} \textbf{conv}\ C &\subseteq \textbf{conic}\ C \cap \textbf{aff}\ C \\ &= \begin{cases} \textbf{conic}\ C, & \text{if } \mathbf{0} \in \textbf{aff}\ C \quad \text{(i)} \\ \textbf{conv}\ C, & \text{otherwise.} \quad \text{(ii)} \end{cases} \end{aligned} \tag{5}$$

Q7 .... (total: 14 Points)

Let $f(\mathbf{X}) = \text{rank}(\mathbf{X})$, with $\textbf{dom}\ f \triangleq \mathcal{B} = \{\mathbf{X} \in \mathbb{R}^{M\times N} \mid \|\mathbf{X}\|_* = \|\mathbf{X}\|_{\mathcal{A}} \leq 1\} = \textbf{conv}\ \mathcal{A}$, where $\mathcal{A}$ is the set of rank-1 matrices. Prove that the convex envelope of f can be shown to be

$$g_f(\mathbf{X}) = \|\mathbf{X}\|_* = \sum_{i=1}^{\text{rank}(\mathbf{X})} \sigma_i(\mathbf{X}), \quad \mathbf{X} \in \mathcal{B}.$$

Hint

You might find it helpful to refer to [1, Eq. (3.61)].

Q8 .... (total: 10 Points)

Prove that a function $f : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex *if and only if* for all $\mathbf{x} \in \textbf{dom}\ f$ and for all $\mathbf{v}$, the function $g(t) = f(\mathbf{x} + t\mathbf{v})$ is quasiconvex on its domain $\{t | \mathbf{x} + t\mathbf{v} \in \textbf{dom}\ f\}$ ([1, Remark 3.31]).

Q9 .... (total: 10 Points)

Let $K \subseteq \mathbb{R}^m$ be a proper cone and $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$. Prove that $\boldsymbol{f}(\mathbf{x})$ is K-convex *if, and only if,* the associated epigraph defined as

$$\textbf{epi}_K \boldsymbol{f} = \left\{ (\mathbf{x}, \mathbf{t}) \in \mathbb{R}^{n+m} \,\middle|\, \boldsymbol{f}(\mathbf{x}) \preceq_K \mathbf{t} \right\}$$

is a convex set.

References

[1] C.-Y. Chi, W.-C. Li, and C.-H. Lin, *Convex optimization for signal processing and communications: from fundamentals to applications*. CRC Press, 2019, (draft version: 13 June 2019).