

Homework #1
Coverage: Chapter 1–3
Due date: 29 March, 2019

Instructor: Chong-Yung Chi

TAs: Amin Jalili, Yi-Wei Li & Ping-Rui Chiang

Notice:

1. Please hand in your answer sheets by yourself to TAs in the class time or to the WCSP Lab., EECS building, R706, before 23:59 of the due date. **No late homework will be accepted.**
2. This homework includes **7 Problems** in **14 pages** with **100 points**.
3. Please justify your answers with clear, logical and solid reasoning or proofs.
4. You need to **print** the Problem Set and answer the problems in the **blank boxes** after each problem or sub-problem. We provided enough space for every problem. However, if you need more space, you can print it in one-side manner (each page in one side of an A4), and use the back side as an additional space.
5. Please do the homework independently by yourself. However, you may discuss with someone else but copied homework is not allowed. This will show your **respect toward the academic integrity**.
6. Write your name, student ID, email and department on the beginning of your answer sheets.
7. Your **legible handwriting** is fine. However, you are very welcome to use text formatting packages for writing your answers.

| | |
|---------------|--|
| Name | |
| Student ID | |
| Department | |
| Email Address | |

| Problem | Score |
|---------|-------|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| Total | |

Problem 1. (35 points) Which of the following statements is TRUE and which one is FALSE? Justify your answers.

(i) Let $V \triangleq \{\mathbf{x} = (x_1, x_2, x_3) \mid \mathbf{x} \in \mathbb{R}^3\}$. Let $\mathbf{y} = (y_1, y_2, y_3)$, $\mathbf{w} = (w_1, w_2, w_3) \in V$, $t \in \mathbb{R}$ and consider

$$\mathbf{y} + \mathbf{w} \triangleq (y_1 + w_1, y_2 + 2w_2, y_3 - 3w_3),$$

$$t\mathbf{y} \triangleq (ty_1, ty_2, ty_3).$$

Then V is a subspace.

(ii) The set \mathcal{A} of all 2×2 lower triangular matrices forms a subspace for the space $\mathbb{R}^{2 \times 2}$.

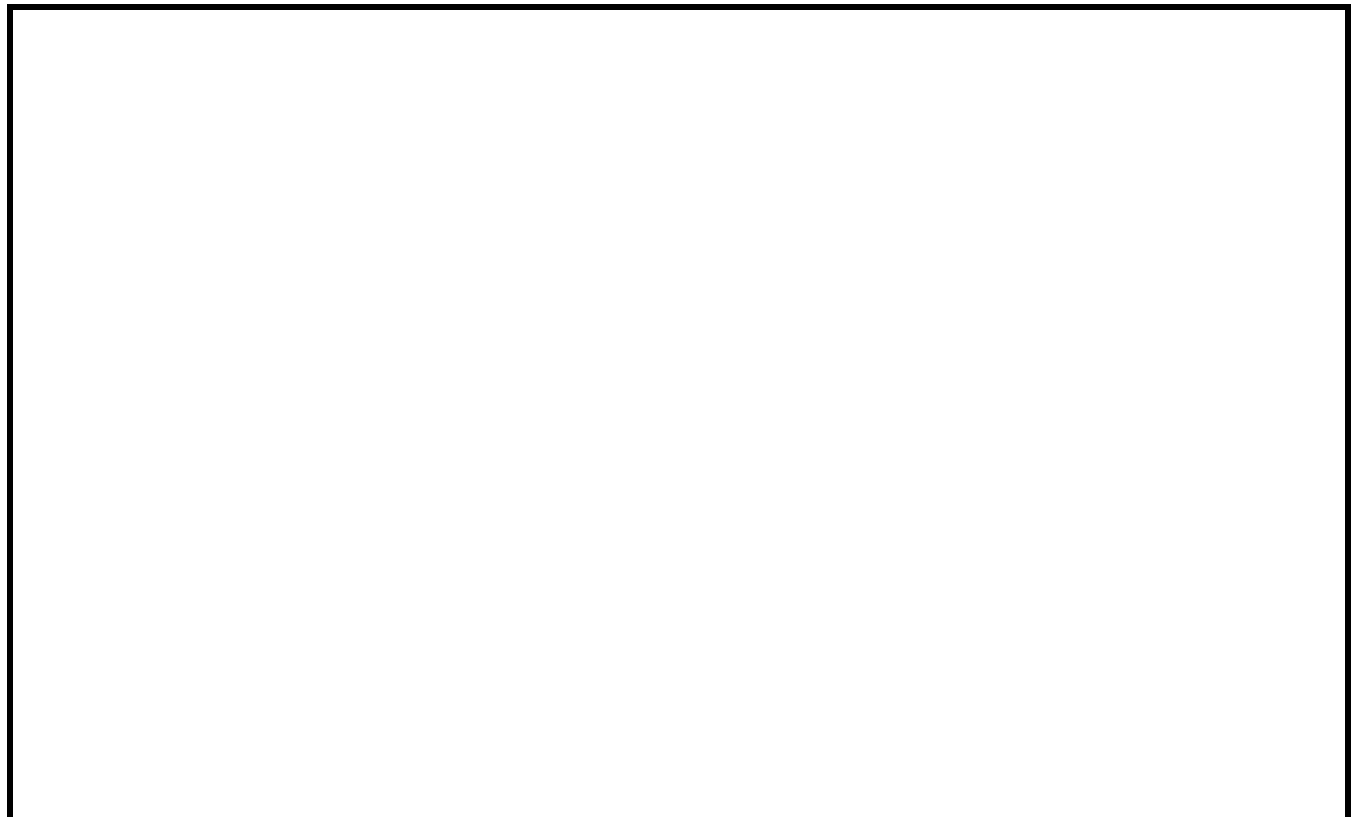
- (iii) Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$. Suppose the system $\mathbf{A}\mathbf{x} = \mathbf{0}_n$ has infinitely many solutions and $\mathbf{B}\mathbf{x} = \mathbf{0}_n$ has one solution. This implies the system $\mathbf{A}\mathbf{B}\mathbf{x} = \mathbf{0}_n$ has exactly one solution.

- (iv) Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. If \mathbf{x} is in the nullspace of \mathbf{A} , then \mathbf{x} is in the nullspace of \mathbf{A}^2 .

(v) The set $\mathcal{B} \triangleq \{(a, b, c) \in \mathbb{R}^3 \mid a = 4b\}$ is not a subspace of \mathbb{R}^3 .

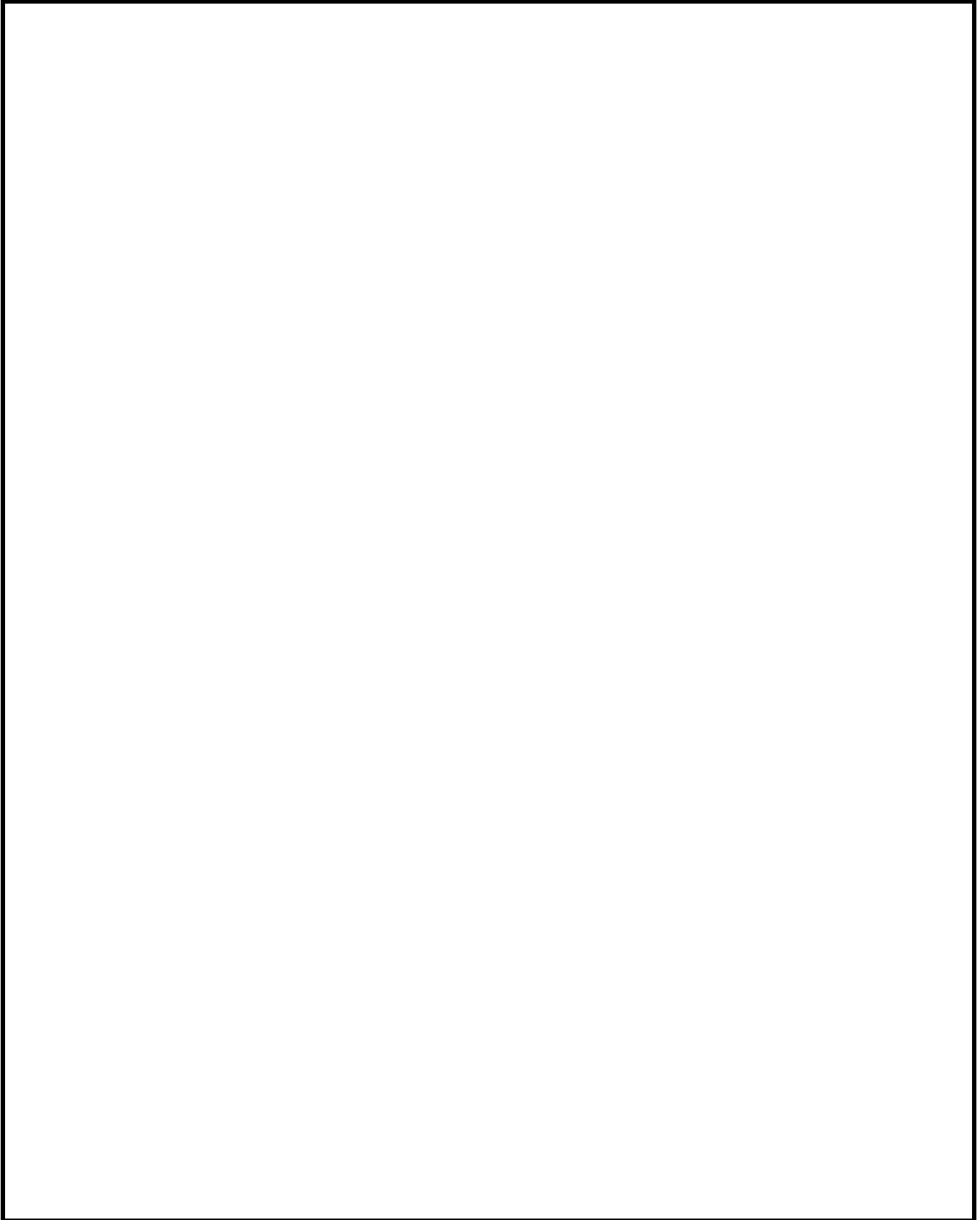


(vi) Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 2 & 6 \end{bmatrix}$. Suppose $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. This linear system of equations is not consistent for any \mathbf{b} (i.e., no solution for any \mathbf{b}).



(vii) Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ and $\mathbf{AB} = \mathbf{I}_n$ where \mathbf{I}_n is the $n \times n$ identity matrix. Then, $\text{rank}(\mathbf{A}) = n$.

Problem 2. (5 points) Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 3 & -3 & -2 & 4 \end{bmatrix}$. Use elimination steps to find the matrix \mathbf{E} such that $\mathbf{EA} = \mathbf{I}$, where \mathbf{I} is the identity matrix.



Problem 3. (10 points) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 8 \\ 7 & 8 & 4 \end{bmatrix}$.

- (i) (5 points) Find a decomposition of \mathbf{A} such that $\mathbf{A} = \mathbf{LU}$ where \mathbf{L} is a lower unitriangular matrix and \mathbf{U} is an upper triangular matrix.

- (ii) (5 points) Consider the system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} = [2 \ 6 \ 19]^T$. Solve this system using the resulting LU decomposition in part (i).

Problem 4. (15 points) Let V be a finite dimensional vector space and $W_1, W_2 \subset V$ be two subspaces with dimensions p and q , respectively.

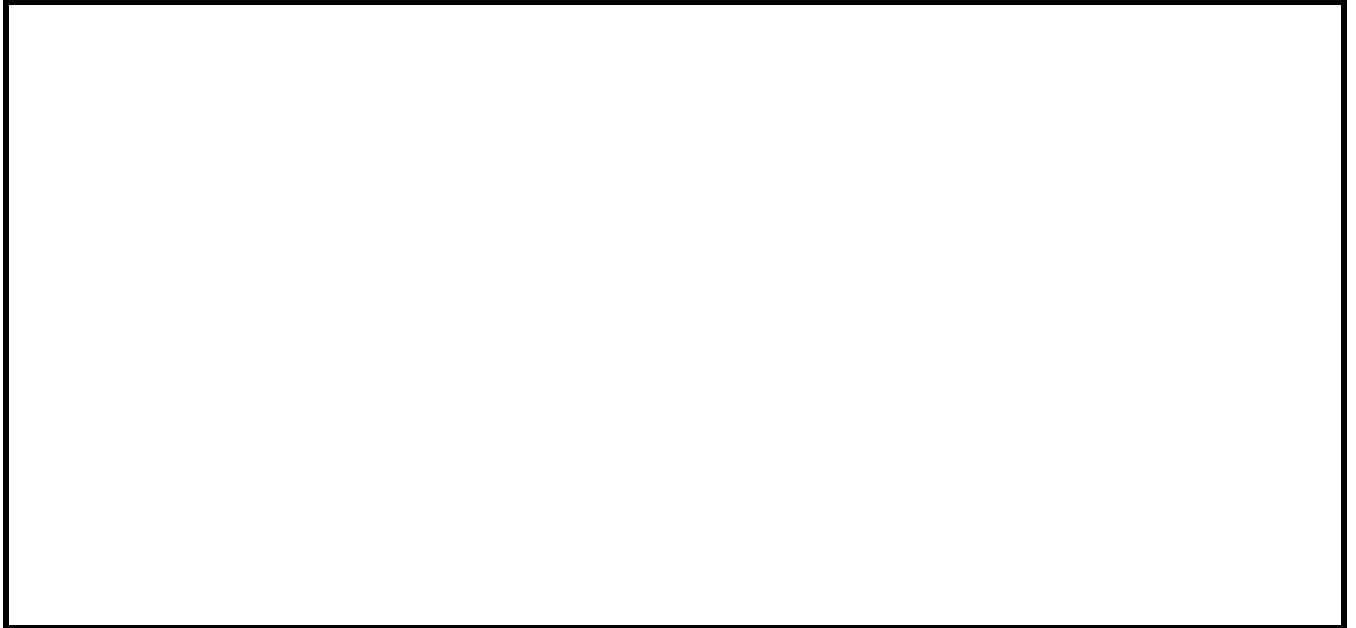
- (i) (5 points) Prove that $W_1 \cap W_2$ is the *largest subspace* of V contained in both W_1 and W_2 .

- (ii) (5 points) Assume $p \geq q$. Show that:

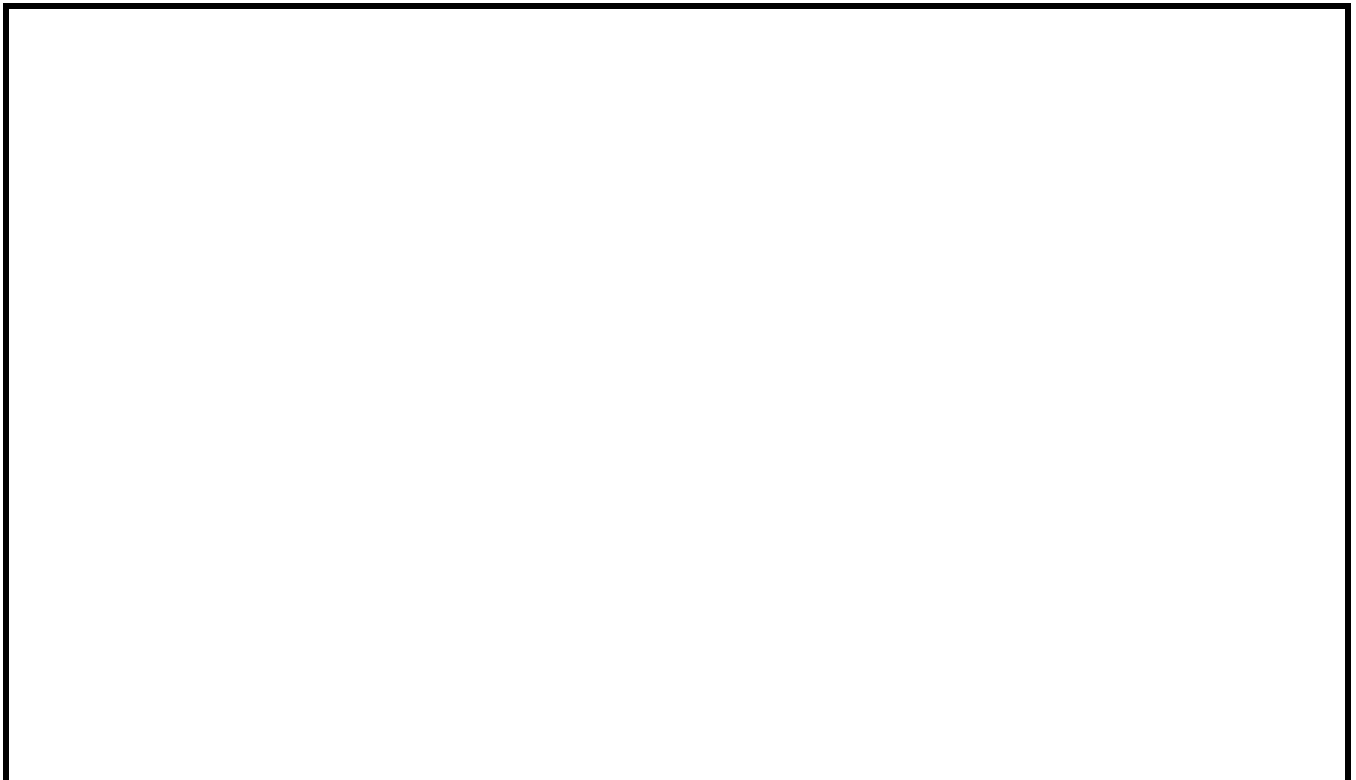
$$\dim(W_1 \cap W_2) \leq q.$$

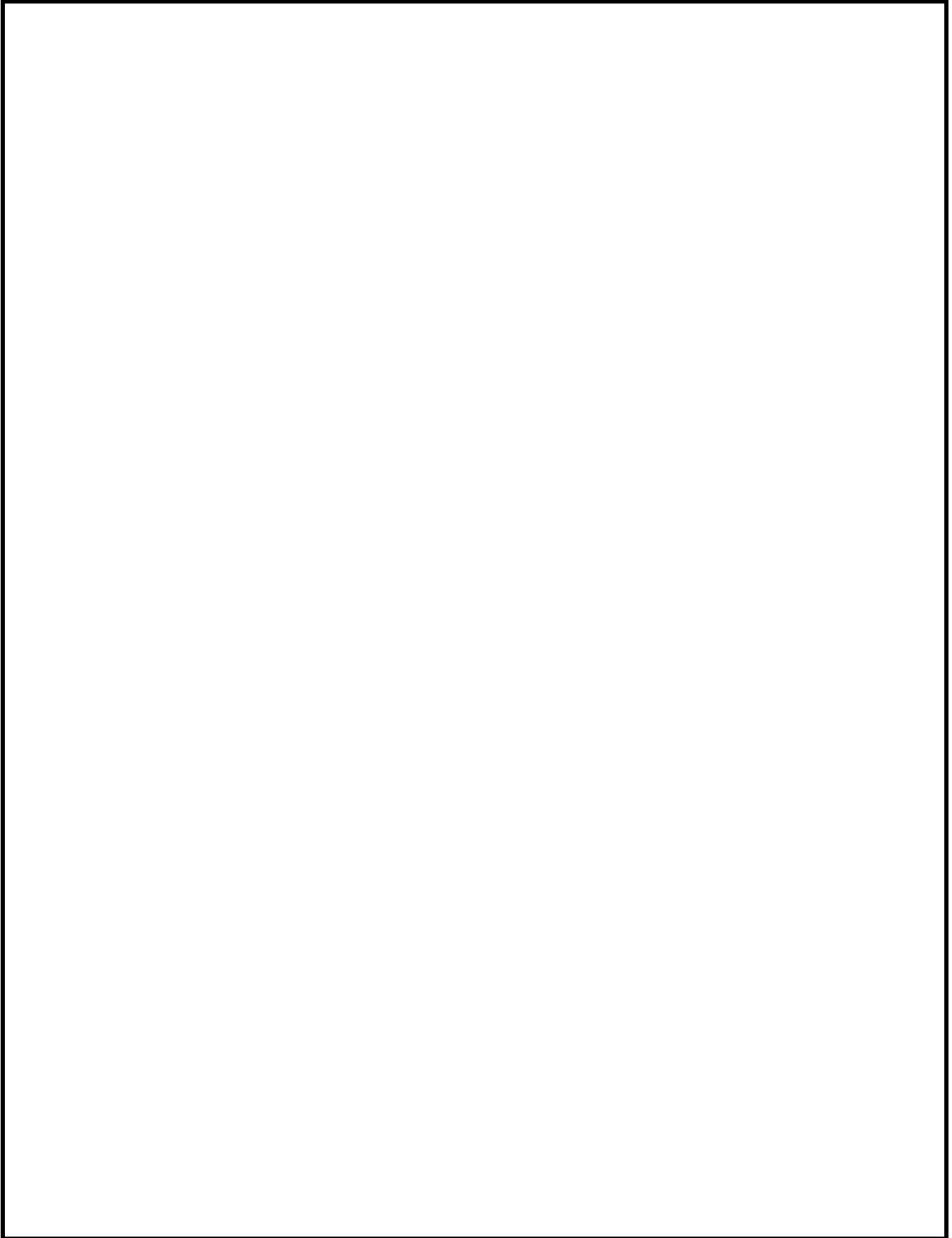
- (iii) (5 points) Let $V = \mathbb{R}^3$ and assume $p > q > 0$. Let $W_1 + W_2 \triangleq \{\mathbf{w}_1 + \mathbf{w}_2 \mid \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}$. Find an example of subspaces W_1 and W_2 such that

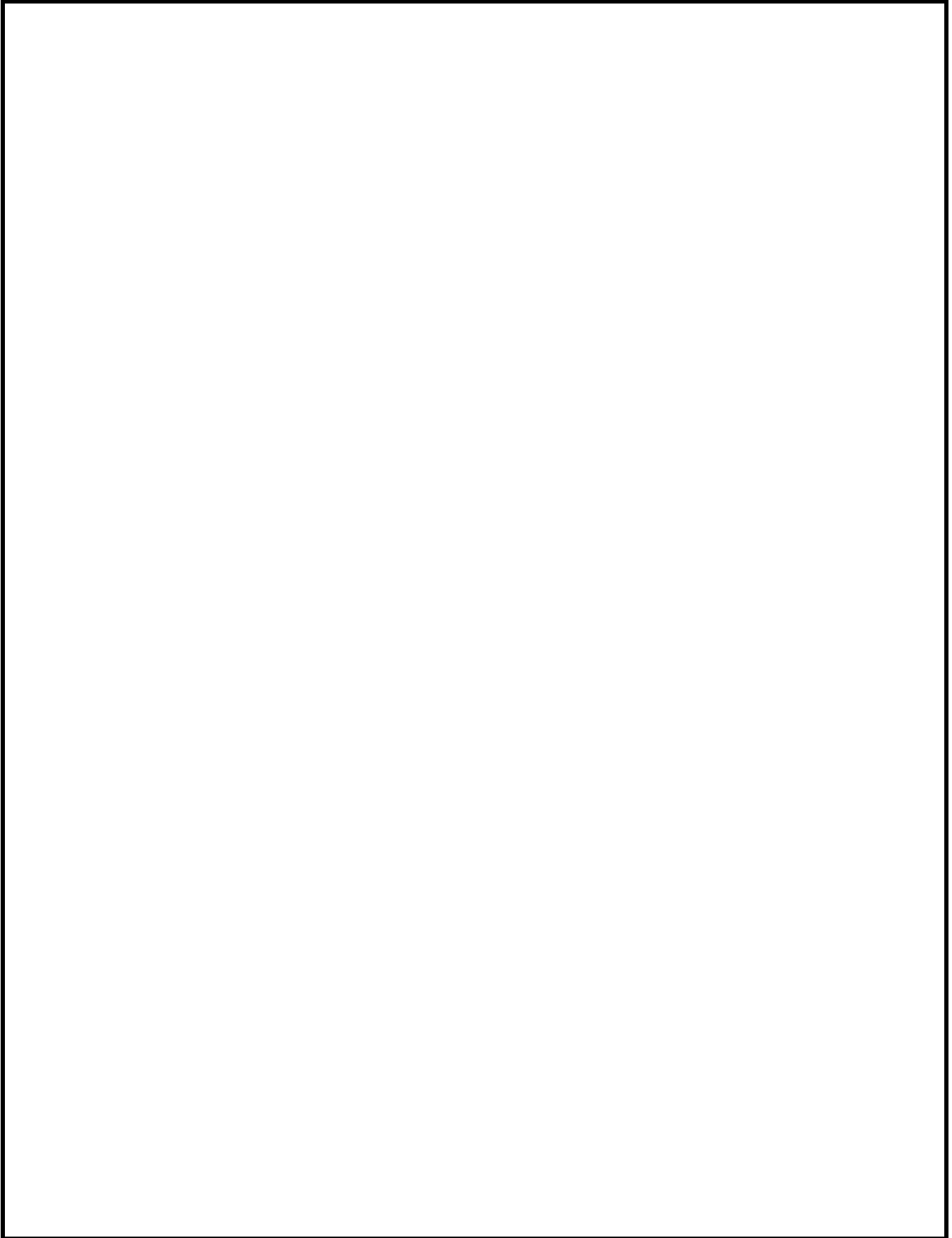
$$\dim(W_1 + W_2) = p + q.$$



Problem 5. (15 points) Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$. Let's denote $C(\cdot)$, $N(\cdot)$, $rank(\cdot)$ as the column space, null space and rank of corresponding matrix, respectively. Find $C(\mathbf{A})$, $N(\mathbf{A})$, $\dim(N(\mathbf{A}))$, $C(\mathbf{A}^T)$, $N(\mathbf{A}^T)$.







Problem 6. (10 points) Let V be a finite dimensional vector space and $W_1, W_2 \subset V$ be two subspaces. Then *sum* of the two subspaces is defined as

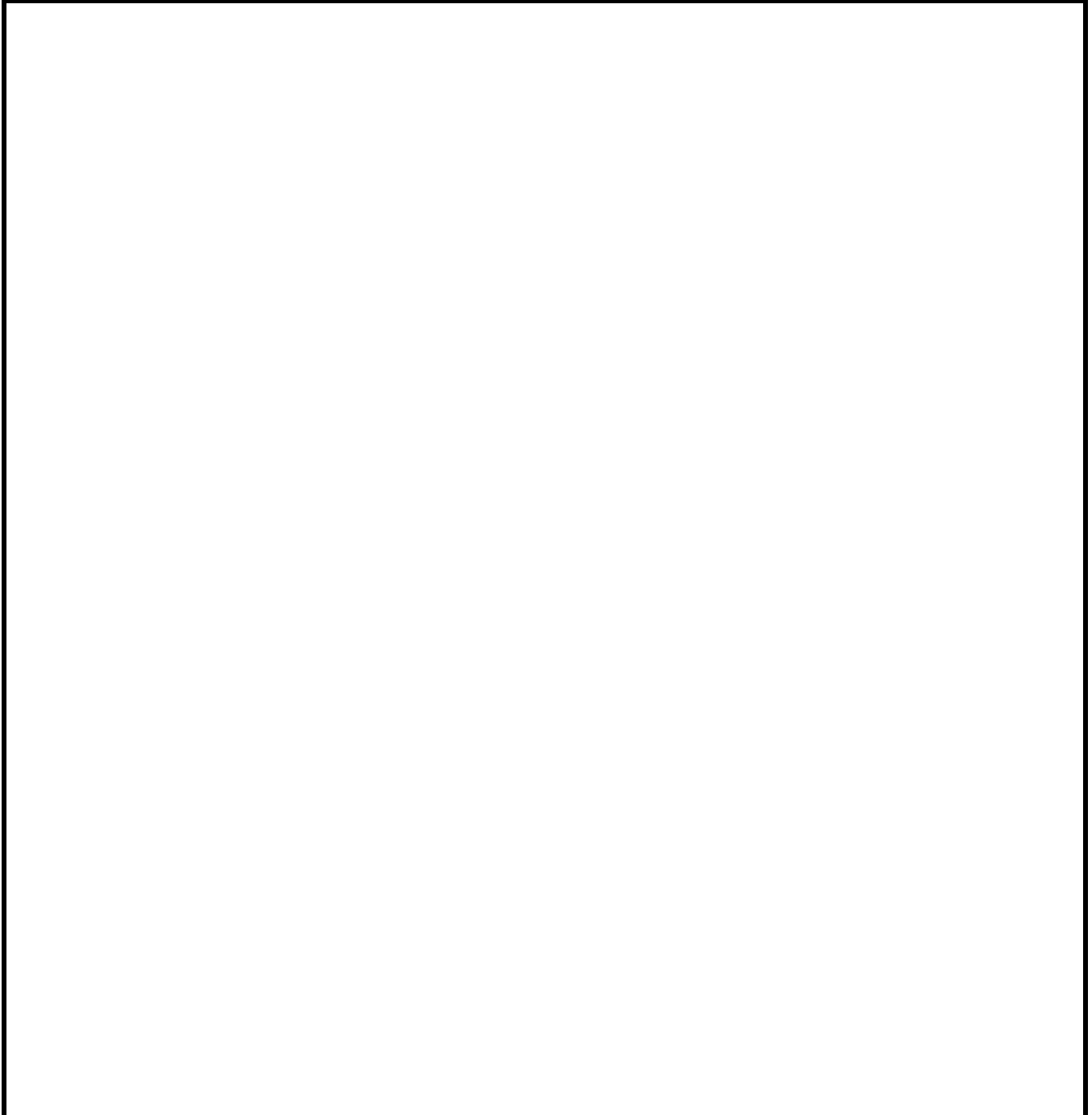
$$S = W_1 + W_2 \triangleq \{\mathbf{w}_1 + \mathbf{w}_2 \mid \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}.$$

Then we define *direct sum* of the two subspaces as

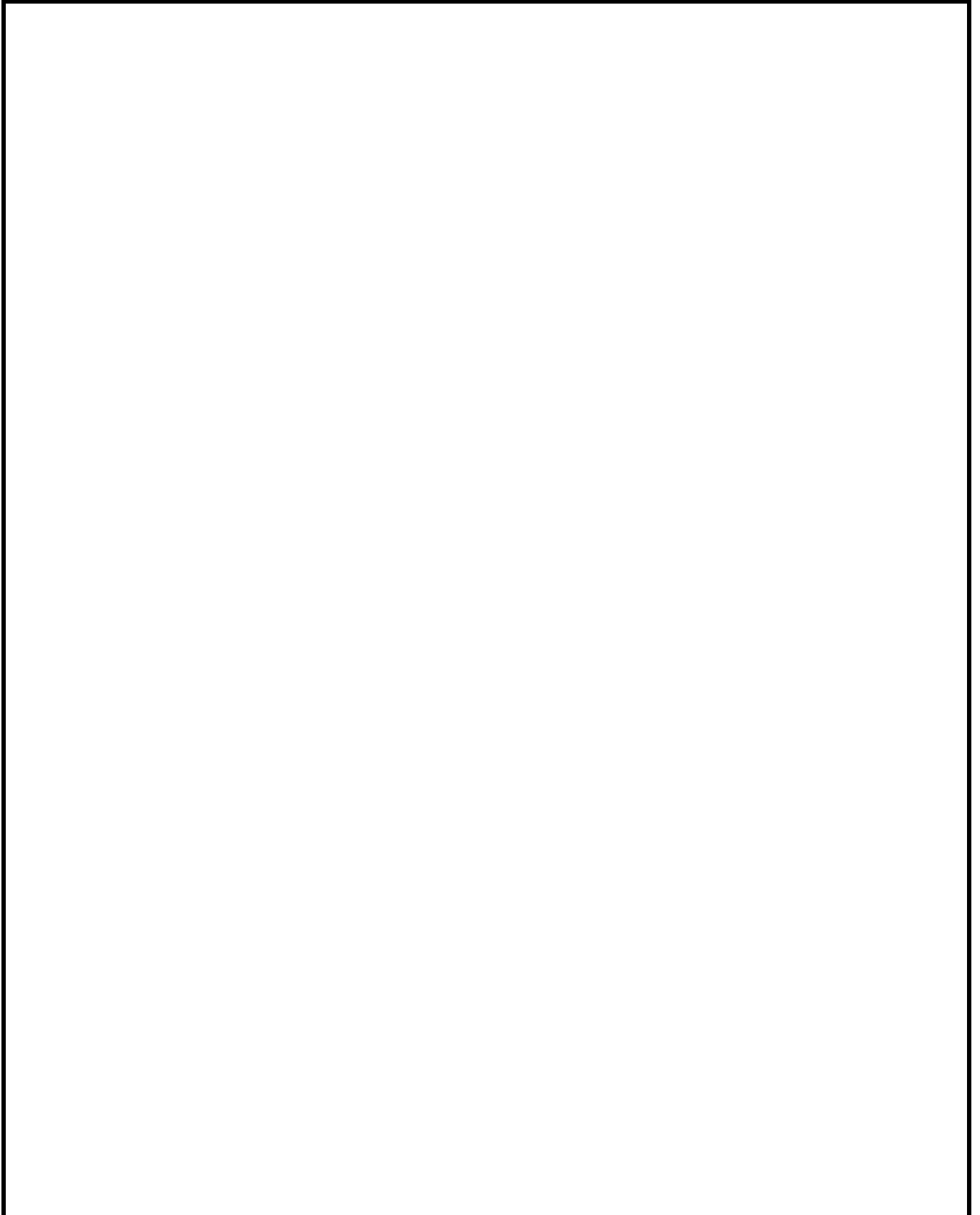
$$\mathcal{S} \triangleq W_1 \oplus W_2,$$

if: (1) $\mathcal{S} = W_1 + W_2$ and (2) $W_1 \cap W_2 = \{\mathbf{0}\}$ (here \oplus accounts for the direct sum).

- (i) (5 points) Prove that $V = W_1 \oplus W_2$ if and only if any vector $\mathbf{v} \in V$ can be uniquely written as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_1 \in W_1$ and $\mathbf{v}_2 \in W_2$.



(ii) (5 points) Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.



Problem 7. (10 points) Let $\mathbf{w}_1 = \begin{bmatrix} 3 \\ 3 \\ 9 \\ 6 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ \beta \\ 3 \\ 2 \end{bmatrix}$ and $\mathbf{w}_3 = \begin{bmatrix} 2 \\ 3 \\ 3\beta \\ 2\beta \end{bmatrix}$ where $\alpha \in \mathbb{R}$.

(i) (5 points) Find the values of β such that \mathbf{w}_1 , \mathbf{w}_2 and \mathbf{w}_3 be linearly independent.

(ii) (5 points) Find the values of β such that $\mathbf{w}_3 \in \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$.