EECS205000: Linear Algebra **Spring 2019**
College of Electrical Engineering and Computer Science
National Tsing Hua University

Homework #1
Coverage: Chapter 1–3
Due date: 29 March, 2019

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Notice:

1. *Please hand in your answer sheets by yourself to TAs in the class time or to the WCSP Lab., EECS building, R706, before 23:59 of the due date.* ***No late homework will be accepted.***
2. *This homework includes* ***7 Problems*** *in* ***14 pages*** *with* ***100 points****.*
3. *Please justify your answers with clear, logical and solid reasoning or proofs.*
4. *You need to* ***print*** *the Problem Set and aanswer the problems in the* ***blank boxes*** *after each problem or sub-problm. We provided enough space for every problem. However, if you need more space, you can print it in one-side manner (each page in one side of an A4), and use the back side as an aditional space.*
5. *Please do the homework independently by yourself. However, you may discuss with someone else but copyied homework is not allowed. This will show your* ***respect toward the academic integrity****.*
6. *Write your name, student ID, email and department on the begining of your ansewr sheets.*
7. *Your* ***legible handwriting*** *is fine. However, you are very welcome to use text formatting packages for writing your answers.*

Name	
Student ID	
Department	
Email Address	

Problem	Score
1	
2	
3	
4	
5	
6	
7	
Total	

Problem 1. (35 points) Which of the following statements is TRUE and which one is FALSE? Justify your answers.

(i) Let $V \triangleq \{\mathbf{x} = (x_1, x_2, x_3) \mid \mathbf{x} \in \mathbb{R}^3\}$. Let $\mathbf{y} = (y_1, y_2, y_3), \mathbf{w} = (w_1, w_2, w_3) \in V$, $t \in \mathbb{R}$ and consider

$$\mathbf{y} + \mathbf{w} \triangleq (y_1 + w_1, y_2 + 2w_2, y_3 - 3w_3),$$
$$t\mathbf{y} \triangleq (ty_1, ty_2, ty_3).$$

Then V is a subspace.

(ii) The set $\mathcal{A}$ of all 2×2 lower triangular matrices forms a subspace for the space $\mathbb{R}^{2\times 2}$.

(iii) Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$. Suppose the system $\mathbf{Ax} = \mathbf{0}_n$ has infinitely many solutions and $\mathbf{Bx} = \mathbf{0}_n$ has one solution. This implies the system $\mathbf{ABx} = \mathbf{0}_n$ has exactly one solution.

(iv) Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. If $\mathbf{x}$ is in the nullspace of $\mathbf{A}$, then $\mathbf{x}$ is in the nullspace of $\mathbf{A}^2$.

(v) The set $\mathcal{B} \triangleq \{(a, b, c) \in \mathbb{R}^3 \mid a = 4b\}$ is not a subspace of $\mathbb{R}^3$.

(vi) Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 2 & 6 \end{bmatrix}$. Suppose $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. This linear system of equations is not consistent for any $\mathbf{b}$ (i.e., no solution for any $\mathbf{b}$).

(vii) Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$ and $\mathbf{AB} = \mathbf{I}_n$ where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Then, $rank(\mathbf{A}) = n$.

Problem 2. **(5 points)** Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 3 & -3 & -2 & 4 \end{bmatrix}$. Use elimination steps to find the matrix $\mathbf{E}$ such that $\mathbf{EA} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

Problem 3. (10 points) Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 8 \\ 7 & 8 & 4 \end{bmatrix}$.

(i) (5 points) Find a decomposition of $\mathbf{A}$ such that $\mathbf{A} = \mathbf{LU}$ where $\mathbf{L}$ is a lower unitriangular matrix and $\mathbf{U}$ is an upper triangular matrix.

(ii) (5 points) Consider the system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{b} = \begin{bmatrix} 2 & 6 & 19 \end{bmatrix}^T$. Solve this system using the resulting LU decomposition in part (i).

Problem 4. (15 points) Let V be a finite dimensional vector space and $W_1, W_2 \subset V$ be two subspaces with dimensions p and q, respectively.

(i) (5 points) Prove that $W_1 \cap W_2$ is the *largest subspace* of V contained in both W_1 and W_2.

(ii) (5 points) Assume $p \geq q$. Show that:

$$dim(W_1 \cap W_2) \leq q.$$

(iii) (5 points) Let $V = \mathbb{R}^3$ and assume $p > q > 0$. Let $W_1 + W_2 \triangleq \{\mathbf{w}_1 + \mathbf{w}_2 \mid \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}$. Find an example of subspaces W_1 and W_2 such that

$$dim(W_1 + W_2) = p + q.$$

Problem 5. (15 points) Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$. Let's denote $C(.)$, $N(.)$, $rank(.)$ as the column space, null space and rank of corresponding matrix, respectively. Find $C(\mathbf{A})$, $N(\mathbf{A})$, $dim(N(\mathbf{A}))$, $C(\mathbf{A}^T)$, $N(\mathbf{A}^T)$.

Problem 6. (10 points) Let V be a finite dimensional vector space and $W_1, W_2 \subset V$ be two subspaces. Then *sum* of the two subspaces is defined as

$$S = W_1 + W_2 \triangleq \{\mathbf{w}_1 + \mathbf{w}_2 \,|\, \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}.$$

Then we define *direct sum* of the two subspaces as

$$\mathcal{S} \triangleq W_1 \oplus W_2,$$

if: (1) $\mathcal{S} = W_1 + W_2$ and (2) $W_1 \cap W_2 = \{\mathbf{0}\}$ (here $\oplus$ accounts for the direct sum).

(i) (5 points) Prove that $V = W_1 \oplus W_2$ if and only if any vector $\mathbf{v} \in V$ can be uniquely written as $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ where $\mathbf{v}_1 \in W_1$ and $\mathbf{v}_2 \in W_2$.

(ii) (5 points) Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Problem 7. (10 points) Let $\mathbf{w}_1 = \begin{bmatrix} 3 \\ 3 \\ 9 \\ 6 \end{bmatrix}$, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ \beta \\ 3 \\ 2 \end{bmatrix}$ and $\mathbf{w}_3 = \begin{bmatrix} 2 \\ 3 \\ 3\beta \\ 2\beta \end{bmatrix}$ where $\alpha \in \mathbb{R}$.

(i) (5 points) Find the values of β such that $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ be linearly independent.

(ii) (5 points) Find the values of β such that $\mathbf{w}_3 \in span\{\mathbf{w}_1, \mathbf{w}_2\}$.