EECS205000: Linear Algebra **Spring 2019**
College of Electrical Engineering and Computer Science
National Tsing Hua University

Homework #2 Solutions
Coverage: Chapter 1–5

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Problem 1. (10 points) Let $\mathbf{A} = \frac{1}{3}\begin{bmatrix} -2 & 2 & \beta_1 \\ 2 & 1 & \beta_2 \\ 1 & 2 & \beta_3 \end{bmatrix}$. Is it possible to find values of $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ such that $\mathbf{A}$ to be an orthogonal matrix? Justify your answer by rigorous reasoning.

Solution:

We know $\mathbf{A}$ is orthogonal if and only if its columns provide an orthonormal set of vectors. Then we have

$$\begin{cases} -2\beta_1 + 2\beta_2 + \beta_3 = 0 \\ 2\beta_1 + \beta_2 + 2\beta_3 = 0 \end{cases},$$

which clearly implies

$$\mathbf{B}\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $\mathbf{B} = \begin{bmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ and hence $\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix}^T \in N(\mathbf{B})$. The reduced row echelon form (*rref*) of $\mathbf{B}$ can be obtained as

$$rref(\mathbf{B}) = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \end{bmatrix},$$

and clearly

$$N(\mathbf{B}) = span\left\{\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}\right\} = c\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \quad c \in \mathbb{R}.$$

Since the vector $\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix}^T$ must be of unit length, then $\frac{1}{9}c^2(1^2 + 2^2 + (-2)^2) = 1$ and hence $c = \pm 1$. Clearly,

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \pm\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

Problem 2. (5 points) Let $\mathbf{b}_1 = [1, 2, 2, 4]^T$, $\mathbf{b}_2 = [-2, 0, -4, 0]^T$, and $\mathbf{b}_3 = [-1, 1, 2, 0]^T$, and let S be the span of these vectors. Apply the Gram-Schmidt process to $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ to obtain an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ for S.

Solution:

Since the given vectors are in $\mathbb{R}^4$, the norm of the a vector $\mathbf{x}$ can be derived by the following formula:

$$\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

First step,

$$\mathbf{v}_1 = \mathbf{b}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{5}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 4 \end{bmatrix}$$

Next step,

$$\mathbf{v}_2 = \mathbf{b}_2 - \frac{\mathbf{b}_2^T\mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = \frac{4}{5}\begin{bmatrix} -2 \\ 1 \\ -4 \\ 2 \end{bmatrix}, \quad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{5}\begin{bmatrix} -2 \\ 1 \\ -4 \\ 2 \end{bmatrix}$$

Finally step,

$$\mathbf{v}_3 = \mathbf{b}_3 - \frac{\mathbf{b}_3^T\mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{b}_3^T\mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = \frac{2}{5}\begin{bmatrix} -4 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{5}\begin{bmatrix} -4 \\ 2 \\ 2 \\ -1 \end{bmatrix},$$

Therefore, $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ is an orthonormal basis for S.

Problem 3. (20 points) Let $\mathbf{A}$ be an $m \times n$ matrix.

(a) (5 points) Show that $N(\mathbf{A}^T\mathbf{A})$=$N(\mathbf{A})$.

(b) (5 points) Show that $rank(\mathbf{A}^T\mathbf{A}) = rank(\mathbf{A})$. (Hint: you may use the rank-nullity theorem, i.e. $dim(N(\mathbf{A})) + rank(\mathbf{A}) = n$ for any $m \times n$ matrix $\mathbf{A}$.)

(c) (10 points) If $\mathbf{A}$ is a 4×3 matrix and $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Let $\tilde{\mathbf{x}}$ be a least-squares solution that minimizes $\|\mathbf{b} - \mathbf{A}\mathbf{x}\|^2$ for $\mathbf{b} = [0, 2, 1, -1]^T$. Find $\mathbf{p} = \mathbf{A}\tilde{\mathbf{x}}$ and give its physical meaning.

Solution:

(a) First, we show that $N(\mathbf{A}) \subseteq N(\mathbf{A}^T\mathbf{A})$. Let $\mathbf{x} \in N(\mathbf{A})$, then $\mathbf{A}\mathbf{x} = \mathbf{0}$, this implies $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{0} = \mathbf{0}$ for all $\mathbf{x}$. Therefore, $\mathbf{x} \in N(\mathbf{A}^T\mathbf{A})$ and hence $N(\mathbf{A}) \subseteq N(\mathbf{A}^T\mathbf{A})$. Next we show that $N(\mathbf{A}^T\mathbf{A}) \subseteq N(\mathbf{A})$. Let $\mathbf{x} \in N(\mathbf{A}^T\mathbf{A})$, then $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{0}$, this implies $\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = (\mathbf{A}\mathbf{x})^T\mathbf{A}\mathbf{x}$=$\mathbf{0}$. Clearly, $\mathbf{A}\mathbf{x} = \mathbf{0}$ and hence $\mathbf{x} \in N(\mathbf{A})$. So, $N(\mathbf{A}^T\mathbf{A}) \subseteq N(\mathbf{A})$. Therefore, $N(\mathbf{A}^T\mathbf{A})$=$N(\mathbf{A})$.

(b) The $N(\mathbf{A}^T\mathbf{A}) = N(\mathbf{A})$ implies $dim(N(\mathbf{A}^T\mathbf{A}))$= dim $(N(\mathbf{A}))$. By the rank-nullity theorem, we know that $dim(N(\mathbf{A})) + rank(\mathbf{A}) = n$ and also $dim(N(\mathbf{A}^T\mathbf{A})) + rank(\mathbf{A}^T\mathbf{A}) = n$. Therefore, $rank(\mathbf{A}^T\mathbf{A}) = rank(\mathbf{A})$.

(c) First, $\mathbf{p} = \mathbf{A}\tilde{\mathbf{x}}$ is the projection of $\mathbf{b}$ onto $C(\mathbf{A})$. To find $\tilde{\mathbf{x}}$ we must solve the equation $\mathbf{A}^T\mathbf{A}\tilde{\mathbf{x}} = \mathbf{A}^T\mathbf{b}$. However, since the first and third columns of $\mathbf{A}$ are the same, while the first and second columns are linear independent. This means $\mathbf{A}$ only has fist two linear independent columns and we can simplify our calculations by equivalently using the basis matrix $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$, due to $C(\mathbf{B}) = C(\mathbf{A})$, and we can solve the equation $\mathbf{B}^T\mathbf{B}\tilde{\mathbf{x}} = \mathbf{B}^T\mathbf{b}$ to find $\mathbf{p} = \mathbf{B}\tilde{\mathbf{x}}$. So that we can derive

$$\mathbf{B}^T\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix}$$

and

$$\mathbf{B}^T\mathbf{b} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

The equation is then

$$\begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix} \tilde{\mathbf{x}} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow \tilde{\mathbf{x}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Finally, the $\mathbf{p}$ can be derived by

$$\mathbf{p} = \mathbf{B}\tilde{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

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Problem 4. (10 points) Assume that the matrix set $\mathbf{M}$ consists of 2×2 real matrices to form a vector space over $\mathbb{R}^{2\times 2}$.

(a) (5 points) Show that the subspace W consisting of symmetric matrices is a subspace of $\mathbf{M}$.

(b) (5 points) Find a basis for W and determine the dimension of W.

Solution:

(a) Let $\mathbf{W}_1$, $\mathbf{W}_2$ be any symmetric matrices in $\mathbf{M}$. We need to show that $\mathbf{0}^{2\times 2}$, $\mathbf{W}_1 + \mathbf{W}_2$ and $c\mathbf{W}_1$ are symmetric. First of all, it is apparent that the $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is symmetric. Moreover, if $\mathbf{W}_1 = \begin{bmatrix} x_1 & y_1 \\ y_1 & z_1 \end{bmatrix}$, $\mathbf{W}_2 = \begin{bmatrix} x_2 & y_2 \\ y_2 & z_2 \end{bmatrix}$, then

$$c\mathbf{W}_1 = \begin{bmatrix} cx_1 & cy_1 \\ cy_1 & cz_1 \end{bmatrix}, \mathbf{W}_1 + \mathbf{W}_2 = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ y_1 + y_2 & z_1 + z_2 \end{bmatrix}$$

is also symmetric. So W is a subspace.

(b) Let the following matrices,

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

be a basis for W. For $a_1, a_2, a_3 \in \mathbb{R}$, the linear combination is:

$$a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$$

which is equal to $\mathbf{0}$ if and only if $a_1 = a_2 = a_3 = 0$. Moreover, since any symmetric matrix $\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}$ can be represented as a linear combination by

$$\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, W has a basis of three elements, and it has dimension 3.

Problem 5. (5 points) Let $\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$. Find the QR decomposition such that $\mathbf{A} = \mathbf{QR}$ where $\mathbf{Q}$ and $\mathbf{R}$ are the orthogonal and upper triangular matrices, respectively.

Solution:

By Gram-Schmidt orthogonalization we will decompose it such that $\mathbf{A} = \mathbf{QR}$ where $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3]$ is orthogonal and $\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$. Let $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$. Then we can write

$$\mathbf{v}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{q}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad r_{11} = \mathbf{q}_1^T \mathbf{a}_1 = 2.$$

Then,

$$\mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2^T \mathbf{v}_1}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = \begin{bmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{bmatrix}, \quad \mathbf{q}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \quad r_{12} = \mathbf{q}_1^T \mathbf{a}_2 = 3, \ r_{22} = \mathbf{q}_2^T \mathbf{a}_2 = 5.$$

Finally,

$$\mathbf{v}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3^T \mathbf{v}_1}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\mathbf{a}_3^T \mathbf{v}_2}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}, \quad \mathbf{q}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix},$$

$$r_{13} = \mathbf{q}_1^T \mathbf{a}_3 = 2, \ r_{23} = \mathbf{q}_2^T \mathbf{a}_3 = -2, r_{33} = \mathbf{q}_3^T \mathbf{a}_3 = 4.$$

Then we have

$$\mathbf{Q} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}.$$

Problem 6. (10 points) Let $y = r + sx^2$, where $r, s \in \mathbb{R}$, provide the least squares fit to the points $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (2, 4)$ and $(x_3, y_3) = (4, 8)$.

(a) (5 points) Find r and s.

(b) (5 points) Find values of y_1, y_2 and y_3 at $x_1 = 1$, $x_2 = 2$ and $x_3 = 4$, respectively, such that the best fitting curve is $y = 0$.

Solution:

(a) The three equation corresponding to these points are

$$\begin{aligned} r + s &= 1 \\ r + 4s &= 4 \\ r + 16s &= 8, \end{aligned}$$

and then we can write

$$\mathbf{A}^T \mathbf{A} \hat{\mathbf{x}} = \mathbf{A}^T \mathbf{b}$$

where $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \\ 1 & 16 \end{bmatrix}$, $\hat{\mathbf{x}} = [r \ \ s]^T$ and $\mathbf{b} = [1 \ \ 4 \ \ 8]^T$. Then we have $\mathbf{A}^T \mathbf{A} = \begin{bmatrix} 3 & 21 \\ 21 & 273 \end{bmatrix}$ and

$$\begin{bmatrix} 3 & 21 \\ 21 & 273 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 13 \\ 145 \end{bmatrix} \Rightarrow r = \frac{4}{3}, s = \frac{3}{7}.$$

(b) If the $y = 0$, it means that $\mathbf{A}^T\mathbf{b} = \mathbf{0}$ and this implies $\mathbf{b}$ is in the null space of $\mathbf{A}^T$. The basis for $N(\mathbf{A}^T)$ is

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \mathbf{b} = k \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}, \quad k \in \mathbb{R}.$$

Problem 7. (20 points) Let $\mathbf{A} \in \mathbb{R}^{3\times 5}$. Assume that we performed row operations on $\mathbf{A}$ to convert it to *rref* form, but now we do something different - instead of getting the usual $\mathbf{R} = [\mathbf{I} \quad \mathbf{F}]$, we now reduce it to a matrix in the form of $\tilde{\mathbf{A}} = [\mathbf{F} \quad \mathbf{I}]$. And the row operation of $\mathbf{A}$ was given as follows:

$$\mathbf{A} \xrightarrow{rref} \begin{bmatrix} 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 0 & 1 & 0 \\ 6 & 7 & 0 & 0 & 1 \end{bmatrix} = \tilde{\mathbf{A}}$$

(a) (10 points) Find a basis for $N(\mathbf{A})$.

(b) (10 points) Find a matrix $\mathbf{M}$ so that applying the same row elimination matrix associated with $\tilde{\mathbf{A}}$ to $\mathbf{AM}$ can get the usual *rref* form.

$$\mathbf{AM} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 4 & 5 \\ 0 & 0 & 1 & 6 & 7 \end{bmatrix}$$

Solution:

(a) Note that row operations always preserve the null space $N(\mathbf{A})$, that is, any solution to $\mathbf{Ax} = \mathbf{0}$ will be preserved by row operations. We can still seek special solutions to $\tilde{\mathbf{A}}\mathbf{x} = \mathbf{0}$ using the usual method. Columns 3, 4, 5 are the pivot columns, while columns 1 and 2 are the free columns. Therefore, we look for row special solutions:

$$\mathbf{s}_1 = \begin{bmatrix} 1 \\ 0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 0 \\ 1 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

We can then see that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ -7 \end{bmatrix}$, i.e. the negative entries of each column of $\mathbf{F}$.
This gives us a basis for the null space of $\mathbf{A}$:

$$N(\mathbf{A}) = span \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ -5 \\ -7 \end{bmatrix} \right\}.$$

(b) We want to first reorder the columns of $\tilde{\mathbf{A}}$ so that it is in the usual *rref* form. Recall that column operations are equivalent to multiplying on the right by an appropriate matrix. A matrix that will put the columns of $\tilde{\mathbf{A}}$ in the correct order is the following permutation matrix

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

The matrix $\mathbf{R}= \tilde{\mathbf{A}}\mathbf{M}$ will then be in the usual *rref* form. Remember that we performed different row operations to put our matrix $\mathbf{A}$ into the different form $\tilde{\mathbf{A}}$, and row operations won't change the column order. In particular, recalled that row operations are equivalent to multiplying by an appropriate matrix on the left, so there exists a matrix $\mathbf{E}$ so that $\mathbf{EA}=\tilde{\mathbf{A}}$. Then $\mathbf{R}=\tilde{\mathbf{A}}\mathbf{M}=\mathbf{EAM}=\mathbf{E}(\mathbf{AM})$ is the usual *rref* form. So performing the same row operations on the $\mathbf{AM}$ will give us a matrix in the usual *rref* form.

Problem 8. (10 points) Answer the following questions.

(a) (5 points) Let $\mathbf{A}=\begin{bmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{bmatrix}$. Find $det(\mathbf{A})$ in terms of x, y, z.

(b) (5 points) Let matrix $\mathbf{A}=[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$, where $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are vectors in $\mathbb{R}^3$. If $4\mathbf{a}_1-3\mathbf{a}_2+2\mathbf{a}_3=\mathbf{0}$, find $det(\mathbf{A})$.

Solution:

(a) By row elimination we have

$$\begin{bmatrix}1 & x & x^2\\ 1 & y & y^2\\ 1 & z & z^2\end{bmatrix}\rightarrow\begin{bmatrix}1 & x & x^2\\ 0 & y-x & y^2-x^2\\ 0 & z-x & z^2-x^2\end{bmatrix}\rightarrow(y-x)(z-x)\begin{bmatrix}1 & x & x^2\\ 0 & y-x & y^2-x^2\\ 0 & z-x & z^2-x^2\end{bmatrix}\rightarrow(y-x)(z-x)\begin{bmatrix}1 & x & x^2\\ 0 & 1 & y+x\\ 0 & 1 & z+x\end{bmatrix}$$

$$\rightarrow(y-x)(z-x)\begin{bmatrix}1 & x & x^2\\ 0 & 1 & y+x\\ 0 & 0 & z-y\end{bmatrix}\rightarrow(y-x)(z-x)(z-y)\begin{bmatrix}1 & x & x^2\\ 0 & 1 & y+x\\ 0 & 0 & 1\end{bmatrix},$$

and then $det(\mathbf{A})=(y-x)(z-x)(z-y)$.

(b) The $4\mathbf{a}_1-3\mathbf{a}_2+2\mathbf{a}_3=\mathbf{0}$ can be rewritten as $\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}\begin{bmatrix}4\\-3\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ and this implies that $\begin{bmatrix}4\\-3\\2\end{bmatrix}\in N(\mathbf{A})$. Therefore, $\mathbf{Ax}=\mathbf{0}$ has a non-zero solution and proves that $\mathbf{A}$ is singular. Thus, $det(\mathbf{A})=0$.

Problem 9. (10 points) Let $\mathbf{C}$ be the cofactor matrix of $\mathbf{A}$, and $\mathbf{C}^T=\begin{bmatrix}2 & 1 & 0\\ 4 & 3 & 2\\ -2 & -1 & 2\end{bmatrix}$. Find the $det(\mathbf{A})$ and $\mathbf{A}$. (Hint: you may use $det(c\mathbf{A})=c^n det(\mathbf{A})$) for any constant c and $n\times n$ matrix $\mathbf{A}$).

Solution:

Recall that the formula for $\mathbf{A}^{-1}$ and cofactor matrix in our textbook is $\mathbf{A}^{-1}=\frac{\mathbf{C}^T}{det(\mathbf{A})}$, which implies that $\mathbf{AC}^T=det(\mathbf{A})\mathbf{I}$. Then, we have

$$\begin{aligned}det(\mathbf{AC}^T)&=det(det(\mathbf{A})\mathbf{I})\\ \Rightarrow det(\mathbf{A})det(\mathbf{C}^T)&=(det(\mathbf{A}))^3det(\mathbf{I})\\ \Rightarrow det(\mathbf{C}^T)&=(det(\mathbf{A}))^2\end{aligned}$$

Then, the $det(\mathbf{C}^T)$ can be calculated as follows,

$$det(\mathbf{C}^T)=\begin{vmatrix}2 & 1 & 0\\ 4 & 3 & 2\\ -2 & -1 & 2\end{vmatrix}=2\begin{vmatrix}3 & 2\\ -1 & 2\end{vmatrix}-(1)\begin{vmatrix}4 & 2\\ -2 & 2\end{vmatrix}=16-12=4$$

Thus, $det(\mathbf{A})=\pm 2$, and $\mathbf{A}^{-1}$ is

$$\mathbf{A}^{-1}=\frac{\mathbf{C}^T}{det(\mathbf{A})}=\pm\frac{1}{2}\begin{bmatrix}2 & 1 & 0\\ 4 & 3 & 2\\ -2 & -1 & 2\end{bmatrix}=\pm\begin{bmatrix}1 & 1/2 & 0\\ 2 & 3/2 & 1\\ -1 & -1/2 & 1\end{bmatrix}$$

By Gauss-Jordan method,

$$\begin{bmatrix}\mathbf{A^{-1}} & \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3}\end{bmatrix} = \pm \begin{bmatrix} 1 & 1/2 & 0 & 1 & 0 & 0 \\ 2 & 3/2 & 1 & 0 & 1 & 0 \\ -1 & -1/2 & 1 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \pm \begin{bmatrix} 1 & 1/2 & 0 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \pm \begin{bmatrix} 1 & 1/2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -6 & 2 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \longrightarrow \pm \begin{bmatrix} 1 & 0 & 0 & 4 & -1 & 1 \\ 0 & 1 & 0 & -6 & 2 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Therefore, the $\mathbf{A} = \pm \begin{bmatrix} 4 & -1 & 1 \\ -6 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}$.