EE367000

Introduction to Mathematics for Communications: Convex Analysis and Optimization

Homework 1

Due: 27 March 2019 (23:59)

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NOTE

- Hand in your hard-copy no later than the due date (ECE 706).
- As a NTHU student, strong academic ethics is assumed and punished otherwise by deducting the whole homework score.

Name: ____________

id: ____________

Total points: 100 — 7 Questions

Q1 .... (total: 15 Points)

By considering [1, sections 1.1.1-3], let the two vectors $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$,

(a) Prove this lower bound for difference of any two vectors. (5 pt.)

$$\|\mathbf{x}-\mathbf{y}\| \geq \Big| \|\mathbf{x}\| - \|\mathbf{y}\| \Big|$$

(b) Show that for any $p > 1$, (10 pt.)

$$\|\mathbf{x}+\mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p.$$

Hint

Inequalities are quite important in convex analysis and you'll see more use of them throughout the course. Here, you might need to use some of these inequalities presented in the book [1, Section 1.1.1-3].

Q2 .... (total: 10 Points)

Consider $\mathbf{A} \in \mathbb{R}^{n\times n}$, $a \geq 1$ and $\mathbf{A}^{-1}$ exists. Referring to induced norms discussed in the book ([1, p.7]), prove that for any $\mathbf{A}$:

$$\|\mathbf{A}\|_a = \frac{1}{\min\limits_{\|\mathbf{u}\|_a=1} \|\mathbf{A}^{-1}\mathbf{u}\|_a}.$$

Hint

Note that $\frac{1}{\min f_+} = \max \frac{1}{f_+}$. In this special relation, note that the denominator is always positive (f_+) and you can reformulate the original term as such. Since you're working with norm functions here, this is the case and this helps simplify the proof. You can also take a look at [1, Section 4.1.3].

Q3 .... (total: 15 Points)

Based on the definitions provided in [1, Section 1.1.6] on interior, closure and boundary, discuss these questions:

(5 pt.) (a) Consider the set $\mathcal{S}_1 = \{(a,b) \in \mathbb{R}^2,\ ab \geq 0\}$ and $\mathcal{S}_2 = \{(a,b) \in \mathbb{R}^2,\ a \neq b\}$, argue whether these two sets are closed or open and write their interior and boundary sets.

(10 pt.) (b) Prove that $\mathbf{bd}\,\mathcal{S} = \emptyset$ is necessary and sufficient (*i.e.* $\Leftrightarrow$) for $\mathcal{S}$ to be both close and open at the same time.

Q4 .... (total: 15 Points)

(10 pt.) (a) Consider the vector space $\boldsymbol{W}$ with two subspaces of $\boldsymbol{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ and $\boldsymbol{V} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{v}\}$. Prove:

$$\operatorname{span}[\boldsymbol{U}] = \operatorname{span}[\boldsymbol{V}] \quad \Leftrightarrow \quad \mathbf{v} \in \operatorname{span}[\boldsymbol{U}]$$

(5 pt.) (b) Supposing that $\boldsymbol{W}$ is a subset of vector space $\boldsymbol{V}$, prove that span$[\boldsymbol{W}]$ is the intersection of all the subspaces of $\boldsymbol{V}$ which contains $\boldsymbol{W}$.

Q5 .... (total: 10 Points)

Show that eigenvalues of the matrix $\mathbf{A} = [a_{ij}] \in \mathbb{C}^{n\times n}$ always lie in the union of circles $\mathcal{D}$ defined as:

$$\mathcal{D}_i \triangleq \left\{ z \;\middle|\; |z - a_{ii}| \leq r_i = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}|, \quad i = 1, 2, \ldots, n \right\}.$$

Q6 .... (total: 15 Points)

Considering the [1, Section 1.2.5-7], prove the following statements:

(5 pt.) (a) All eigenvalues of a Hermitian matrix, $\mathbf{A} \in \mathbb{H}^n$, are real.

(5 pt.) (b) In the case of a Hermitian matrix, considered in part (a), all eigenvectors associated with distinct eigenvalues of $\mathbf{A}$ are orthogonal.

(5 pt.) (c) The singular values and eigenvalues of a matrix $\mathbf{B} \in \mathbb{S}^n_{++}$ are the same.

Q7 .... (total: 20 Points)

(10 pt.) (a) Find the determinant of the unitary matrix $\mathbf{U}_{n\times n}$ ($\mathbf{U}^H\mathbf{U} = \mathbf{I}$). (Note that the matrix $\mathbf{U}$, is complex valued and "^{H}" indicates the Hermitian (i.e. conjugate transpose) of the matrix.).

Question 7 may continues on the next page...

(b) Show that for the matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\det(\mathbf{A}^H\mathbf{A}) > 0$ if and only if $\operatorname{rank}\mathbf{A} = n$. (What about the equality case?) (10 pt.)

A short discussion on proofs

Throughout this course you'll encounter numerous proofs either in the book or homework. Proof is a step by step [chain of] logical reasons arguing the truth of a statement. By trying to prove a mathematical statement you'll construct/show your understanding of the subject. Take the point where you stuck in the process of constructing your chain of reasons as a good sign which shows where you need to fill the gap in your understanding. You can back and forth between the book (or other resources) to fill the gaps in your comprehension.

Generally proofs are usually done by one of these schemes:

- **Direct Proof:** By proving $A \Rightarrow Z$ directly, you're actually construct your proof from the assumption A: assuming A is `TRUE` and use A to show that Z is `TRUE` as well. This usually happens by some intermediate steps (*i.e.* $A \Rightarrow B \Rightarrow \cdots \Rightarrow Z$).
- **Contrapositive Proof:** This is actually another form of direct proof but instead of proving it from A to Z, it proves the opposite way by assuming the opposite of Z (*i.e.* $\neg Z$). Now you can construct your chain of reasoning to find $\neg A$ (*i.e.* $\neg Z \Rightarrow \cdots \Rightarrow \neg A$). Note that $A \Rightarrow Z$ is equivalent of $\neg Z \Rightarrow \neg A$.
- **Proof by Contradiction:** In prove by contradiction you assume not only A but also $\neg Z$ and then show a contradiction by this assumption which proves your initial assumption *aka* $(A, \neg Z)$ is `FALSE`. You know A is `TRUE` so $\neg Z$ is `FALSE`, so Z is `TRUE`.
- **Proof by Induction:** Imagine you want to prove yourself that all the birds on our campus could sing (just as a simple analogy). One pretty straight-forward way to prove is to study all the birds which live on campus and see them singing. We as human are often quick to get frustrated and come to a general conclusion that all birds on our campus sing. This is hard and cumbersome in reality but sometimes in mathematics this way of proof come quite handy *esp.* when you have an ordered set. First you set a basis for your proof (*e.g.* prove for $n = 1$ that the statement is `TRUE`). Having the basis, you **suppose** the statement is `TRUE` for $n = k$. Now, prove it for $n = k + 1$. This way you're proving the desired statement (*e.g.* singing) for all the member of your set (*e.g.* birds).

References

[1] Chong-Yung Chi, Wei-Chiang Li, and Chia-Hsiang Lin. *Convex optimization for signal processing and communications: from fundamentals to applications*. CRC Press, 2017.