EE367000

Introduction to Mathematics for Communications:

Convex Analysis and Optimization

Homework 3

Due: 06 June 2019 (12:05 PM)

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NOTE

- You can hand in your hard-copy in the class or at our lab located in ECE building, room 706.
- Submissions later than the due date will be accepted with a point deduction (5% for each hour).
- As a NTHU student, strong academic ethics is assumed and punished otherwise by deducting the whole homework score.

Name: ______________________

id: ______________________

Total points: 100 11 Questions

Q1 .... (total: 9 Points)

(a) The set C_1 is a convex set with non-empty interior. Let C_2 be a non-empty convex set that $C_2 \cap \textbf{int}\, C_1 = \emptyset$. Show that there exists a hyperplane, $\mathcal{H}$, such that $\mathcal{H}_+$ (or $\mathcal{H}_-$) contains C_2, and does not intersect with the interior of C_1. (5 pt.)

(b) By a counter example, disprove similar statement as part (a) but instead of $C_2 \cap \textbf{int}\, C_1 = \emptyset$, consider $C_2 \cap \textbf{relint}\, C_1 = \emptyset$. (4 pt.)

Q2 .... (total: 8 Points)

Consider $f : I \to \mathbb{R}$ as a convex function and $I \subset \mathbb{R}$. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be two sequences of real numbers in I. Suppose these three properties:

$$(P1) \qquad a_1 \geq a_2 \geq \cdots \geq a_n, \quad b_1 \geq b_2 \geq \cdots \geq b_n,$$

$$(P2) \qquad \sum_{i=1}^{r} a_i \geq \sum_{i=1}^{r} b_i \quad (\forall r < n),$$

$$(P3) \qquad \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \, .$$

Prove the below inequality,

$$\sum_{i=1}^{n} f(a_i) \geq \sum_{i=1}^{n} f(b_i).$$

Q3 .... (total: 4 Points)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ be a convex and continuous function. Prove,

$$\mathcal{G}_f(\mathbf{x}) = \{\overline{\nabla} f(\mathbf{x})\} \neq \emptyset, \quad \forall \mathbf{x} \in \mathbb{R}^n,$$

where $\mathcal{G}_f(\mathbf{x})$ denotes the set of all the subgradients of $f(\mathbf{x})$ at $\mathbf{x}$.

Q4 .... (total: 8 Points)

Consider $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$. Show that the function $f : \mathbb{R}^n \to \mathbb{R}$,

$$f(\mathbf{x}) = \begin{cases} -(x_1 x_2 \dots x_n)^{\frac{1}{n}}, & \text{if } x_i > 0,\ i = 1, 2, \dots, n \\ \infty, & \text{otherwise} \end{cases},$$

is convex.

Q5 .... (total: 10 Points)

A function f is called log-convex if $f(\mathbf{x}) > 0$, $\forall \mathbf{x} \in \mathbf{dom}\, f$ and $\ln\big(f(\mathbf{x})\big)$ be convex.

(5 pt.) (a) Show that if f is log-convex, then f is convex.

(5 pt.) (b) Show that f is log-convex if and only if

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq f(\mathbf{x})^{\theta} f(\mathbf{y})^{1-\theta}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbf{dom}\, f,$$

where $\forall \theta \in [0, 1]$.

Q6 .... (total: 7 Points)

Let g be a convex function. Consider the function f given by

$$f(x) = \int_{-\infty}^{\infty} g(t) \exp^{-(\frac{x-t}{\sigma})^2} dt,$$

where $x \in \mathbb{R}$ and σ are scalar. Show that f is a convex function.

Q7 .... (total: 11 Points)

(3 pt.) (a) Consider f as a strictly quasiconvex function on the convex set P. Show that,

$$\forall p_1, p_2 \in P, \quad f(p_1) < f\Big(\frac{p_1 + p_2}{2}\Big) \Rightarrow f\Big(\frac{p_1 + p_2}{2}\Big) < f(p_2).$$

(8 pt.) (b) Suppose Q be a compact set and P be a convex set and let $f(q, p)$ be defined on $Q \times P$ such that:

1. $\forall q \in Q$, the function $f(q, p)$ is strictly quasiconvex on P,
2. $\forall p \in P$, the function $f(q, p)$ is continuous function on Q.

Consider the function $g(p) = \max_{q \in Q} f(q, p)$, prove g is strictly quasiconvex on P (You may use the result in part (a) to prove this statement).

Q8 .... (total: 11 Points)

Let $K \subseteq \mathbb{R}^m$ be a proper convex cone with associated generalized inequality $\preceq_K$, and let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$. For $\boldsymbol{\alpha} \in \mathbb{R}^m$, the $\boldsymbol{\alpha}$-sublevel set of $\mathbf{f}$ (with respect to $\preceq_K$) is defined as

$$S_{\boldsymbol{\alpha}} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \preceq_K \boldsymbol{\alpha}\}.$$

The epigraph of $\mathbf{f}$, with respect to $\preceq_K$, is defined as the set

$$\mathbf{epi}_K \mathbf{f} = \{(\mathbf{x}, \mathbf{t}) \in \mathbb{R}^{n+m} \mid \mathbf{f}(\mathbf{x}) \preceq_K \mathbf{t}\}.$$

Show the followings:

(a) If $\mathbf{f}$ is K-convex, then its sublevel sets $S_{\boldsymbol{\alpha}}$ are convex for all $\boldsymbol{\alpha}$. (4 pt.)

(b) The function $\mathbf{f}$ is K-convex if and only if $\mathbf{epi}_K \mathbf{f}$ is a convex set. (7 pt.)

Q9 .... (total: 12 Points)

(a) Consider the function $f(\mathbf{X}, t) = nt(\log t) - t \log\det \mathbf{X}$, with $\mathbf{dom}\ f = \mathbb{S}^n_{++} \times \mathbb{R}_{++}$. Show that $f(\mathbf{X}, t)$ is convex. (6 pt.)

(b) Use the result obtained in (a) to show that (6 pt.)

$$g(\mathbf{X}) = n(\mathrm{Tr}(\mathbf{X})) \log(\mathrm{Tr}(\mathbf{X})) - (\mathrm{Tr}(\mathbf{X})) \log\det \mathbf{X},$$

is convex on $\mathbb{S}^n_{++}$.

Q10 .... (total: 16 Points)

(a) Consider the function $f : \mathbb{R} \to \mathbb{R}$ with parameters a, b, c and d defined as $f(x) = ax^3 + bx^2 + cx + d$. Find the range of parameters a, b, c and d for the function f to be *quasiconcave*. (4 pt.)

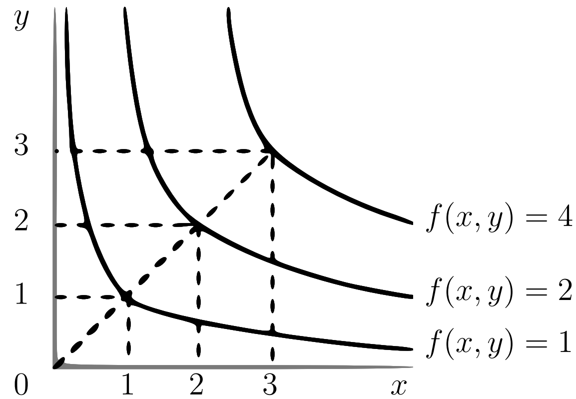
Hint

The [1, Fact 3.3, page 129] may help you solve this question.

(b) Consider two quasiconcave function $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$. Provide an example which $f_1 + f_2$ is not quasiconcave. (3 pt.)

(c) Discuss the part (b) but this time consider f_1 to be concave. (4 pt.)

(d) Figure below depicts the level set representation of the function $f : \mathbb{R}^2 \to \mathbb{R}$. Using information provided in the figure, discuss if the curves are consistent with a notion of concavity and quasiconcavity of f. Note that we do not know the complete information about the function f (e.g. other levels not depicted here). However, you need to discuss the (in)consistency of the limited information provided here regarding (quasi)concavity of function f. (5 pt.)



Q11 ....(total: 4 Points)

Consider $f : \mathbb{R}^{n\times n} \times \mathbb{R}^n \to \mathbb{R}$, defined by,

$$f(\mathbf{X}, \mathbf{y}) = \mathbf{y}^T\mathbf{X}^{-1}\mathbf{y}, \quad \mathbf{dom}\, f = \big\{(\mathbf{X}, \mathbf{y}) \,\big|\, \mathbf{X} + \mathbf{X}^T \succ 0\big\}.$$

The f is convex or not? If it's convex, prove it, and if not, disproves it by a counter example.

References

[1] C.-Y. Chi, W.-C. Li, and C.-H. Lin, *Convex optimization for signal processing and communications: from fundamentals to applications*. CRC Press, 2017.

[2] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.