EE367000

Introduction to Mathematics for Communications:
Convex Analysis and Optimization

Homework 2

Due: 09 May 2019 (12:05 PM)

Instructor: Prof. Chong-Yung Chi

TAs: Sadid Sahami & PingRui Chiang

NOTE

- You can hand in your hard-copy in the class or at our lab located in ECE building, room 706.
- Submissions later than the due date will be accepted with a point deduction (5% for each hour).
- As a NTHU student, strong academic ethics is assumed and punished otherwise by deducting the whole homework score.

Name: ________________

id: ________________

Total points: 100 — 11 Questions

Q1 .... (total: 5 Points)

Consider the two matrices $\mathbf{A}$ and $\mathbf{B}$ with the same number of rows. Let $\textbf{conic}\,\mathbf{A}$ denote the conic hull of columns of $\mathbf{A}$. Prove there exists $\boldsymbol{P} \succeq \mathbf{0}$ such that $\mathbf{B} = \mathbf{A}\boldsymbol{P}$ *if and only if* $\textbf{conic}\,\mathbf{B} \subseteq \textbf{conic}\,\mathbf{A}$, where "$\succeq$" stands for componentwise inequality.

Q2 .... (total: 16 Points)

(a) Let $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{N+2}\}$ be a set of points ($\mathbf{a}_j \in \mathbb{R}^N$, $j = 1, 2, \ldots, N+2$, $N \in \mathbb{N}$) and all the points are distinct. Prove that for any A, there are always two subsets, B and C ($B \neq C$), such that: (8 pt.)

$$\textbf{conv}\,B \cap \textbf{conv}\,C \neq \emptyset.$$

Hint

Considering $\mathbf{b}_j \triangleq (\mathbf{a}_j, 1)$ might help you.

(b) Consider each of $A_1, A_2, \ldots, A_k$ as a convex set of points in $\mathbb{R}^N$ where $k > N$. Prove that if the $\bigcap_{l \in \mathcal{I}_{N+1}} A_l \neq \emptyset$ for any $\mathcal{I}_{N+1}$, then, (8 pt.)

$$\bigcap_{l=1}^{k} A_l \neq \emptyset.$$

Here, $\mathcal{I}_{N+1}$, is a subset of $\{1, 2, \ldots, k\}$ with $N+1$ elements.

Hint

You may need to use the result of part (a) in part (b).

Q3 .... (total: 8 Points)

Prove that the set Q,

$$Q = \left\{ \mathbf{Diag}\left(\mathbf{A}\right) - \mathbf{A} \circ \mathbf{A} \,\middle|\, \mathbf{A}^2 = \mathbf{A},\ \mathbf{A} \in \mathbb{S}^n \right\},$$

is a subset of $\mathbb{S}^n_+$. Then show whether the set Q is a convex cone or not. Here, $\mathbf{Diag}\left(\mathbf{A}\right)$ is a diagonal matrix with just the diagonal elements of $\mathbf{A}$. The operation "$\circ$" denotes the Hadamard matrix multiplication (entrywise product, $[\mathbf{A} \circ \mathbf{B}]_{ij} = [\mathbf{A}]_{ij}[\mathbf{B}]_{ij}$).

Q4 .... (total: 12 Points)

(8 pt.) (a) Let K be a convex cone. Prove K is pointed *if and only if* the origin, $\mathbf{0}$, is an extreme point of K.

(4 pt.) (b) Give an example of non-pointed <u>convex</u> cone.

Q5 .... (total: 8 Points)

Let $\tilde{\mathbf{x}} \in P \triangleq \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{A}\mathbf{x} \preceq \boldsymbol{b}, \quad \mathbf{A} \in \mathbb{R}^{m \times n}, \boldsymbol{b} \in \mathbb{R}^m\}$, where $\operatorname{rank}\left(\mathbf{A}\right) = n$. Consider $\tilde{\mathbf{A}}, \tilde{\boldsymbol{b}}$ as subsets of rows of $\mathbf{A}, \boldsymbol{b}$ in which the equality holds ($\tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\boldsymbol{b}}$). Prove $\tilde{\mathbf{x}}$ is an extreme point of P *if and only if* $\operatorname{rank}(\tilde{\mathbf{A}}) = n$.

Q6 .... (total: 13 Points)

(8 pt.) (a) Suppose that C and D are closed convex cones in $\mathbb{R}^n$ and C^* and D^* are the associated dual cones. Show that

$$(C \cap D)^* = C^* + D^*.$$

(5 pt.) (b) Let $V = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}\mathbf{x} \succeq \mathbf{0}\}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{A} \neq \mathbf{0}_{m \times n}$. Show that the dual cone of V can be expressed as

$$V^* = \{\mathbf{A}^T\mathbf{v} | \mathbf{v} \succeq \mathbf{0}\} \subset \mathbb{R}^n.$$

Hint

You may need to use the result of part (a).

Q7 .... (total: 8 Points)

Consider three n-dimensional sets $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C} \subseteq \mathbb{R}^n$, where $\mathcal{A}$ is a closed set. Assume that $\mathcal{H}$ is a supporting hyperplane of both $\mathcal{A}$ and $\mathcal{C}$. Prove that $\mathbf{bd}\,\mathcal{A} \cap \mathcal{H} \subseteq \mathbf{bd}\,\mathcal{B}$.

Q8 .... (total: 9 Points)

The dual norm of $\|\cdot\|$ on $\mathbb{R}^n$ is defined as

$$\|\mathbf{x}\|_* \triangleq \sup\left\{ \mathbf{x}^T\mathbf{y} \,\middle|\, \|\mathbf{y}\| \leq 1,\, \mathbf{y} \in \mathbb{R}^n \right\}.$$

Show that $\|\cdot\|_*$ satisfies:

Question 8 may continues on the next page...

(a) Positive definiteness (3 pt.)

(b) Positive homogeneity (3 pt.)

(c) Triangle inequality (3 pt.)

Q9 .... (total: 5 Points)

Let $\mathcal{H}$ be a supporting hyperplane of the convex set $C \in \mathbb{R}^n$. Consider the set $D \triangleq C \cap \mathcal{H}$, prove that $D_{\text{extr}} \subseteq C_{\text{extr}}$. Here, $D_{\text{extr}}, C_{\text{extr}}$ denote the set of extreme points of the sets D and C, respectively.

Q10 .... (total: 8 Points)

Consider two nonempty interior convex cones, K_1 and K_2. Show that if $\textbf{int}\, K_1 \cap \textbf{int}\, K_2 = \emptyset$, there is a $\mathbf{y} \neq \mathbf{0}$ such that $\mathbf{y} \in K_1^*$, $-\mathbf{y} \in K_2^*$.

Q11 .... (total: 8 Points)

Let two non-empty sets, K and C, be subsets of $\mathbb{R}^n$. At least one of them is a cone and $\textbf{cl}\, K \cap \textbf{cl}\, C \neq \emptyset$. Consider a hyperplane, $\mathcal{H}(\mathbf{a} \in \mathbb{R}^n, b)$, which separates K and C such that,

$$b = \sup \left\{ \mathbf{a}^T \mathbf{x} | \mathbf{x} \in K \right\}.$$

Prove that the hyperplane passes through the origin.

References

[1] Chong-Yung Chi, Wei-Chiang Li, and Chia-Hsiang Lin. *Convex optimization for signal processing and communications: from fundamentals to applications.* CRC Press, 2017.