Approximation Bounds for Semidefinite Relaxation of Max-Min-Fair Multicast Transmit Beamforming Problem

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***Abstract*—Consider a downlink multicast scenario where a base station equipped with multiple antennas wishes to simultaneously broadcast a number of signals to some given groups of users over a common bandwidth. The goal of the base station is to select appropriate beamforming vectors so as to maximize the minimum signal-to-interference-plus-noise ratio (SINR) among all users under a power budget constraint. Since this max-min-fair transmit beamforming problem is NP-hard in general, a randomized polynomial time approximation approach based on semidefinite relaxation (SDR) has been proposed recently where excellent performance in both simulated and measured wireless channels has been reported. This paper shows that the SDR-based approach can provide at least an $\mathcal{O}(1/M)$ approximation to the optimum solution, where M is the total number of users. This estimate implies that the SDR solution achieves an SINR that is at most $(\log M + \mathcal{O}(1))$ dB away from the highest possible value. The existence of such a data independent bound certifies the worst-case approximation quality of the SDR algorithm for any problem instance and any number of transmit antennas. For real-valued problems, the corresponding approximation ratio is shown to be $\mathcal{O}(1/M^2)$, while the SINR loss due to SDR approximation is at most $(2\log M + \mathcal{O}(1))$ dB.**

***Index Terms*—Approximation bound, multicast, semidefinite relaxation (SDR), transmit beamforming.**

I. Introduction

TRADITIONAL methods for broadcasting rely on a single antenna to radiate signal power isotropically in space, and use nonoverlapping spectrum to separate user signals. Recently, transmit beamforming for multicast has become a subject of great interest [1]–[8]. In this approach, a transmitter (e.g., a base station) employs multiple antennas to form appropriate spatial beampatterns and uses them to simultaneously broadcast multiple signals over a common frequency band. The beampatterns are carefully designed to reduce co-channel interference between different groups of receivers so as to achieve a desired quality of service (QoS) at each receiver. In this manner, the base station can communicate with multiple receivers simultaneously over a common frequency band, achieving higher spectrum efficiency.

For the max-min-fair transmit beamformer [1], [8]–[10], the beampatterns are designed such that the minimum signal-to-interference-plus-noise ratio (SINR) at receivers is maximized under a power constraint at the base station. The max-min-fair transmit beamforming problem is nonconvex and has been shown [1], [8] to be NP-hard in general. Thus, we are naturally led to high-quality approximate solutions for this problem which are efficiently computable in polynomial time. In the single-group multicast scenario where SINR reduces to signal-to-noise ratio (SNR), an approximation method based on iterative diagonalization was proposed in [4]. This method, however, requires the assumption that the number of receivers is no more than the number of transmit antennas. Moreover, this method cannot guarantee the quality of the approximate solutions nor an upper bound on its computational complexity. An alternative convex relaxation approach was introduced in [8] for this problem. It was found that a good approximate solution can be obtained by a semidefinite relaxation (SDR)-based approach. The latter is a powerful technique to tackle nonconvex optimization problems and has been successfully applied to several other signal processing problems; see [11]–[13] and references therein. The SDR approach requires solving a semidefinite program (SDP) followed by a simple randomization procedure to obtain a feasible solution, all of which can be completed in polynomial time. It has been proven in [14] that, the SDR-based approach provides a worst-case $\mathcal{O}(1/m_1)$ approximation quality for the *single-group* multicast problem, where m_1 denotes the total number of receivers.

In the paper, we focus on the *multigroup* multicast transmit beamforming problem and study the worst-case performance of the associated SDR approach. The SDR approach for this multigroup multicast scenario was proposed in [1], where extensive simulations were carried out to demonstrate the effectiveness of this approach. Unlike the single group multicast case which involves the max-min of a set of convex quadratic functions (defined by SNR), the multigroup multicast formulation involves max-min of a set of *fractional* convex quadratic functions, with denominators signifying the interference between groups. The presence of the interference term significantly complicates the performance analysis of the SDR approach. In the paper, we

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present an analysis to show that the SDR-based approach can provide at least an $\mathcal{O}(1/M)$ approximation to the otherwise difficult-to-compute globally maximum min-SINR, where M denotes the total number of receivers among all groups. This result implies that the SDR solution achieves a min-SINR that is at most $(\log M + \mathcal{O}(1))$ dB away from the highest possible value. The existence of such a data independent bound certifies the worst-case approximation quality achievable by the SDR algorithm for any problem instance and any number of transmit antennas. For real-valued problems, we also show that the corresponding SDR-based approach can achieve at least an $\mathcal{O}(1/M^2)$ approximation quality, while the SINR loss due to SDR approximation is at most $(2\log M + \mathcal{O}(1))$ dB. Some simulation results are presented to illustrate the empirical worst-case and average approximation qualities of the SDR-based approach.

II. Multicast Transmit Beamforming via Semidefinite Relaxation

Consider a scenario where a base station equipped with N transmit antennas broadcasts $G (G > 1)$ independent data streams to $M (M = \sum_{k=1}^{G} m_k)$ single-antenna receivers over a common frequency band. Each of the receivers belongs to one of the G groups, with receivers from the same group interested in a common data stream. Let $s_k(t)$ and $\boldsymbol{w}_k \in \mathbb{C}^N$ denote the broadcasting data stream and the transmit weight vector (or beamforming vector) for the kth group, respectively. The transmitted signal at the base station is given by $\sum_{k=1}^{G} \boldsymbol{w}_k^H s_k(t)$. Assume that $s_k(t)$, $k = 1, \ldots, G$, are statistically independent and are temporally white with zero mean and unit variance. Let $\boldsymbol{h}_{k,i} \in \mathbb{C}^{N\times 1}$ denote the channel vector between the base station and the ith receiver in group k. The SINR of the ith receiver in group k is given by

$$\mathrm{SINR}_{k,i} = \frac{\boldsymbol{w}_k^H \mathbf{R}_{k,i} \boldsymbol{w}_k}{\sum_{j\neq k} \boldsymbol{w}_j^H \mathbf{R}_{k,i} \boldsymbol{w}_j + \sigma_{k,i}^2}$$

where $\mathbf{R}_{k,i} = \mathrm{E}\{\boldsymbol{h}_{k,i}\boldsymbol{h}_{k,i}^H\}$ is the channel correlation matrix and $\sigma_{k,i}^2$ is the noise variance.

The idea of transmit beamforming is to design the beamforming weight vectors $\boldsymbol{w}_k$ such that each receiver can retrieve the signal of interest with the desired quality of service (QoS). The QoS is usually measured in terms of SINR. To this end, there are two criteria widely considered in the literature [1]: The QoS-guaranteed transmit beamformer and the max-min-fair transmit beamformer. The QoS-guaranteed transmit beamforming problem is to minimize the transmit power at the base station while making sure the SINR in each receiver is no smaller than some specified value. Mathematically, it can be written as the following optimization problem

$$p^\star = \min_{\substack{\boldsymbol{w}_k \in \mathbb{C}^N \\ k=1,\ldots,G}} \sum_{k=1}^{G} \|\boldsymbol{w}_k\|^2 \tag{1a}$$

$$\text{subject to (s.t.)} \quad \frac{\boldsymbol{w}_k^H \mathbf{R}_{k,i} \boldsymbol{w}_k}{\sum_{j\neq k} \boldsymbol{w}_j^H \mathbf{R}_{k,i} \boldsymbol{w}_j + \sigma_{k,i}^2} \geq \rho_{k,i}, \quad i = 1, \ldots, m_k,\ k = 1, \ldots, G \tag{1b}$$

where $\rho_{k,i} > 0$ is the target SINR value for the ith receiver in group k. The difficulty of the QoS-guaranteed beamforming problem is that, when the target SINR values $\rho_{k,i}$ are set too high or when the number of receivers M is much greater than the number of transmit antennas N, the problem easily becomes infeasible [15]. By contrast, the max-min-fair transmit beamforming problem, which maximizes the minimum SINR value among M receivers subject to the power constraint $P > 0$ at the base station, is always feasible. It can be mathematically formulated as follows:

$$u^\star = \max_{\substack{\boldsymbol{w}_k \in \mathbb{C}^N \\ k=1,\ldots,G}} \min_{\substack{i=1,\ldots,m_k \\ k=1,\ldots,G}} \frac{\boldsymbol{w}_k^H \mathbf{R}_{k,i} \boldsymbol{w}_k}{\sum_{j\neq k} \boldsymbol{w}_j^H \mathbf{R}_{k,i} \boldsymbol{w}_j + \sigma_{k,i}^2} \tag{2a}$$

$$\text{s.t.} \quad \sum_{k=1}^{G} \|\boldsymbol{w}_k\|^2 \leq P. \tag{2b}$$

It can be seen that both problem (1) and problem (2) are nonconvex optimization problems. Moreover, they have been proven [1] to be NP-hard in general.[1] [2] Therefore, an approximation method for obtaining a good approximate solution in polynomial time is desired. In [1], Karipidis *et al.* proposed an approximation method based on SDR. To illustrate this, we define $\mathbf{W}_k = \boldsymbol{w}_k \boldsymbol{w}_k^H$ and rewrite problem (2) as

$$u^\star = \max_{\substack{\mathbf{W}_k \in \mathbb{C}^{N\times N} \\ k=1,\ldots,G}} \min_{\substack{i=1,\ldots,m_k \\ k=1,\ldots,G}} \frac{\mathrm{tr}(\mathbf{R}_{k,i}\mathbf{W}_k)}{\sum_{j\neq k} \mathrm{tr}(\mathbf{R}_{k,i}\mathbf{W}_j) + \sigma_{k,i}^2} \tag{3a}$$

$$\text{s.t.} \quad \sum_{k=1}^{G} \mathrm{tr}(\mathbf{W}_k) \leq P \tag{3b}$$

$$\mathbf{W}_k \succeq \mathbf{0} \ \text{(positive semidefinite)} \tag{3c}$$

$$\mathrm{rank}(\mathbf{W}_k) = 1,\ k = 1, \ldots, G \tag{3d}$$

where $\mathrm{tr}(\cdot)$ stands for the trace of a matrix. By dropping the only nonconvex constraint $(\mathbf{W}_k) = 1$, we obtain the relaxation counterpart of problem (2) as

$$v^\star = \max_{\substack{\mathbf{W}_k \in \mathbb{C}^{N\times N} \\ k=1,\ldots,G}} \min_{\substack{i=1,\ldots,m_k \\ k=1,\ldots,G}} \frac{\mathrm{tr}(\mathbf{R}_{k,i}\mathbf{W}_k)}{\sum_{j\neq k} \mathrm{tr}(\mathbf{R}_{k,i}\mathbf{W}_j) + \sigma_{k,i}^2} \tag{4a}$$

$$\text{s.t.} \quad \sum_{k=1}^{G} \mathrm{tr}(\mathbf{W}_k) \leq P \tag{4b}$$

$$\mathbf{W}_k \succeq \mathbf{0},\ k = 1, \ldots, G. \tag{4c}$$

Note that $v^\star \geq u^\star$ since the feasible set of problem (3) is a subset of that of problem (4). Instead of being NP-hard as problem (2), problem (4) is a quasi-convex optimization problem where the global optimum can be obtained in polynomial time by the bisection algorithm [17] via a sequence of semidefinite feasibility subproblems. The SDR-based approximation method proposed in [1] consists of two stages. In the first stage, problem (4) is solved, while in the second stage, a randomization/multigroup-power-control procedure is applied to the optimal solution of problem (4) for obtaining an approximate solution of problem (2). We refer readers to [1] for further details. In Table I, we present an SDR procedure without involving multigroup power control. Theoretically, the size of random samples L can be chosen as a polynomial of GN in order to guarantee a high-quality approximate solution.

[1]A problem is called NP-hard if and only if the problem is at least as computationally difficult as a known NP-complete problem. Readers may refer to [16] for details about NP-hard problems.

[2]A special scenario for which problem (2) can be solved in polynomial time is investigated in [5].

TABLE I
SDR PROCEDURE TO SOLVE PROBLEM

(S1) Solve (4) using the bisection algorithm and let $\{\mathbf{W}_k^\star\}_{k=1}^G$ be an optimal solution. Let $L > 0$ be an integer.

(S2) For $\ell = 1, 2, ..., L$, generate G random vectors $\boldsymbol{\xi}_k \in \mathbb{C}^N$ from the complex Gaussian distribution $\mathcal{N}_c(\mathbf{0}, \mathbf{W}_k^\star)$, $k = 1, 2, ..., G$, and let

$$(\boldsymbol{w}_1^{(\ell)}, \boldsymbol{w}_2^{(\ell)}, ..., \boldsymbol{w}_G^{(\ell)}) = \frac{\sqrt{P}}{\sqrt{\sum_{k=1}^G \|\boldsymbol{\xi}_k\|^2}}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, ..., \boldsymbol{\xi}_G)$$

and

$$u^{(\ell)} = \min_{\substack{i=1,...,m_k \\ k=1,...,G}} \frac{(\boldsymbol{w}_k^{(\ell)})^H \mathbf{R}_{k,i} \boldsymbol{w}_k^{(\ell)}}{\sum_{j\neq k}(\boldsymbol{w}_k^{(\ell)})^H \mathbf{R}_{k,i} \boldsymbol{w}_k^{(\ell)} + \sigma_{k,i}^2}.$$

(S3) Suppose

$$\ell^\star = \arg \max_{\ell=1,2,...,L} \{u^{(\ell)}\}.$$

Then select $\hat{\boldsymbol{w}}^{(\ell^\star)}$ as the approximate optimum solution of (2), and obtain $\hat{u}^\star = u^{(\ell^\star)}$.

However, in practice, a choice of $L = 30 \sim 50$ suffices. According to [1], the SDR algorithm for (2) in Table I can deliver excellent performance in both simulated and measured wireless channels.

Similarly, the QoS-guaranteed problem in (1) can be relaxed to

$$q^\star = \min_{\substack{\mathbf{W}_k \in \mathbb{C}^{N\times N} \\ k=1,...,G}} \sum_{k=1}^G \mathrm{tr}(\mathbf{W}_k) \tag{5a}$$

$$\text{s.t. } \frac{\mathrm{tr}(\mathbf{R}_{k,i}\mathbf{W}_k)}{\sum_{j\neq k}\mathrm{tr}(\mathbf{R}_{k,i}\mathbf{W}_j)+\sigma_{k,i}^2} \geq \rho_{k,i},\ i=1,\ldots,m_k, \tag{5b}$$

$$\mathbf{W}_k \succeq \mathbf{0},\ k=1,\ldots,G, \tag{5c}$$

and an SDR procedure [1] can be applied for obtaining an approximate solution of problem (1).

III. APPROXIMATION RATIO

Consider the NP-hard max-min-fair transmit beamforming problem (2), as well as its SDR problem (4). Let $\hat{u}^\star$ denote the objective value of the approximate solutions of (2) that is generated by the randomized polynomial time SDR procedure in Table I. Then the approximation quality of the SDR solution can be measured by the ratio of

$$\gamma = \max\{v^\star/u^\star, u^\star/\hat{u}^\star\}.$$

Clearly, the approximation ratio is always greater than or equal to 1. If $\gamma = 1$ for some particular problem instance of (2) and (4), then there is no relaxation gap between (2) and (4), and the corresponding nonconvex problem (2) has been solved to global optimality. If $\gamma > 1$, then the SDR approach generates an approximate solution that is within $(1 - 1/\gamma)$ fraction to the optimum minimum SINR $u^\star$. Empirically, it has been observed [1] that the approximation ratio γ is quite small (in the range of 1$\sim$3) and grows linearly with the number of receivers. The goal of this paper is to theoretically estimate the approximation ratio for the max-min-fair transmit beamforming problem (2). Specifically, we are interested in the worst-case bound which is independent of the channel correlation matrices $\mathbf{R}_{k,i}$, noise variance $\sigma_{k,i}^2$, the number of transmit antennas N, and the power budget P:

$$\gamma_{\mathrm{sdr}} := \max_{\mathbf{R}_{k,i},\sigma_{k,i}^2,N,P} \max\{v^\star/u^\star, u^\star/\hat{u}^\star\}. \tag{6}$$

From its definition, the value of γ_{sdr} directly translates into a guaranteed bound on the SINR performance loss due to SDR approximation: The SINR gap to the maximum achievable is at most $\log \gamma_{\mathrm{sdr}}$ in dB. In general, we cannot compute γ_{sdr} explicitly due to our inability to compute the quantity $u^\star$ in (6) which represents the optimum value of the NP-hard problem (2). Nonetheless, we can provide a tight estimate of the order of γ_{sdr}, as is done in the ensuing analysis.

There has been no prior study of approximation ratios for the multicast transmit beamforming problem under the max-min fairness criterion. All existing analyses have focussed on QoS-guaranteed formulation [14], [18]. However, it is possible to translate some of the existing results to the max-min fairness formulation (2), as we show next.

A. Unicast Scenario: $m_k = 1$

The unicast situation corresponds to the case of each group having exactly one user. Consequently, the multicast transmit beamforming problem (2) becomes the conventional multiuser transmit beamforming problem [9], [10], [18]–[21]:

$$u^\star = \max_{\substack{\boldsymbol{w}_k \in \mathbb{C}^N \\ k=1,...,G}} \min_{k=1,...,G} \frac{\boldsymbol{w}_k^H \mathbf{R}_{k,1} \boldsymbol{w}_k}{\sum_{j\neq k} \boldsymbol{w}_j^H \mathbf{R}_{k,1} \boldsymbol{w}_j + \sigma_{k,1}^2} \tag{7a}$$

$$\text{s.t. } \sum_{k=1}^G \|\boldsymbol{w}_k\|^2 \leq P \tag{7b}$$

and its relaxation counterpart is given by

$$v^\star = \max_{\substack{\mathbf{W}_k \in \mathbb{C}^{N\times N} \\ k=1,...,G}} \min_{k=1,...,G} \frac{\mathrm{tr}(\mathbf{R}_{k,1}\mathbf{W}_k)}{\sum_{j\neq k}\mathrm{tr}(\mathbf{R}_{k,1}\mathbf{W}_j)+\sigma_{k,1}^2} \tag{8a}$$

$$\text{s.t. } \sum_{k=1}^G \mathrm{tr}(\mathbf{W}_k) \leq P, \tag{8b}$$

$$\mathbf{W}_k \succeq \mathbf{0},\ k = 1, \ldots, G. \tag{8c}$$

For this special case, one can show that problem (8) is actually equivalent to problem (7), and hence the corresponding approximation ratio $\gamma_{\mathrm{sdr}} = 1$. This result can be proved by the facts that the max-min-fair problems (7) and (8) are equivalent to the QoS-guaranteed problems (1) and (5) with ρ_k equal to $u^\star$ and $v^\star$, respectively, and the approximation ratio of the QoS-guaranteed problem in the unicast scenario is equal to one [18].

B. Single Group Multicast Scenario: $G = 1$

In the special case of $G = 1$ and $m_1 \geq 1$, the multigroup broadcasting problem reduces to the single group broadcasting problem as studied in [8]. At each receiver, only the additive noise is present which degrades the signal quality. Thus, the QoS is measured by the SNR instead of the SINR. The max-min-fair transmit beamforming problem in this case reduces to

$$u^\star = \max_{\boldsymbol{w}_1 \in \mathbb{C}^N} \min_{i=1,...,m_1} \boldsymbol{w}_1^H \left(\mathbf{R}_{1,i}/\sigma_{1,i}^2\right) \boldsymbol{w}_1 \tag{9a}$$

$$\text{s.t. } \|\boldsymbol{w}_1\|^2 \leq P \tag{9b}$$

and its relaxation counterpart is given by

$$v^\star = \max_{\mathbf{W}_1 \in \mathbb{C}^{N\times N}} \min_{i=1,\ldots,m_1} \operatorname{tr}\left(\left(\mathbf{R}_{1,i}/\sigma_{1,i}^2\right)\mathbf{W}_1\right) \quad (10a)$$
$$\text{s.t. } \operatorname{tr}(\mathbf{W}_1) \le P, \quad (10b)$$
$$\mathbf{W}_1 \succeq \mathbf{0}. \quad (10c)$$

It is easy to verify [8] that, problems (9) and (10) are equivalent up to a scalar to the following two problems, respectively

$$P/u^\star = \min_{\boldsymbol{w}_1\in\mathbb{C}^N} \|\boldsymbol{w}_1\|^2 \quad (11a)$$
$$\text{s.t. } \boldsymbol{w}_1^H\left(\mathbf{R}_{1,i}/\sigma_{1,i}^2\right)\boldsymbol{w}_1 \ge 1,\ i=1,\ldots,m_1, \quad (11b)$$
$$P/v^\star = \min_{\mathbf{W}_1\in\mathbb{C}^{N\times N}} \operatorname{tr}(\mathbf{W}_1) \quad (12a)$$
$$\text{s.t. } \operatorname{tr}\left(\left(\mathbf{R}_{1,i}/\sigma_{1,i}^2\right)\mathbf{W}_1\right) \ge 1,\ i=1,\ldots,m_1, \quad (12b)$$
$$\mathbf{W}_1 \succeq \mathbf{0}. \quad (12c)$$

The approximation bound between problems (11) and (12) has been analyzed in [14] which shows that the ratio $v^\star/u^\star = 1$ for $m_1 \le 3$, and for $m_1 > 3$,

$$1 \le \frac{v^\star}{u^\star} \le 8m_1. \quad (13)$$

The bound in (13) indicates that the SDR-based method can have a worst-case $\mathcal{O}(1/m_1)$ approximation to the optimum solution of (9) for any channel correlation matrices $\mathbf{R}_{1,i}$, noise variance $\sigma_{1,i}^2$, and the number of transmit antennas N. Moreover, a rank-1 approximate solution can be computed in a randomized polynomial time (by a simple randomization procedure) which achieves a minimum SNR of $\hat{u}^\star$ that is within a constant factor of $1/(8m_1)$ to $u^\star$ [14]. Thus, the worst-case approximation ratio $\gamma_{\rm sdr}$ is bounded by $8m_1$.

IV. SDR Approximation Ratio for General Multigroup Multicast

Consider the general multigroup multicast case with $G > 1$ and $m_k \ge 1$ for all $k = 1,\ldots,G$. The corresponding max-min-fair transmit beamforming problem (2) can be written in a more compact form

$$u^\star = \max_{\boldsymbol{w}\in\mathbb{C}^n} \min_{j=1,\ldots,M} \frac{\boldsymbol{w}^H\mathbf{A}_j\boldsymbol{w}}{\boldsymbol{w}^H\mathbf{B}_j\boldsymbol{w}+1} \quad (14a)$$
$$\text{s.t. } \boldsymbol{w}^H\boldsymbol{w} \le P \quad (14b)$$

where $\mathbf{A}_j \ne \mathbf{0}$, $\mathbf{A}_j \succeq \mathbf{0}$ and $\mathbf{B}_j \ne \mathbf{0}$, $\mathbf{B}_j \succeq \mathbf{0}$ for $j = 1,\ldots,M$. This can be seen by defining $\boldsymbol{w} = [\boldsymbol{w}_1^T,\ldots,\boldsymbol{w}_G^T]^T \in \mathbb{C}^n$ with $n = GN$, and by letting $\mathbf{A}_j$ and $\mathbf{B}_j$ be block diagonal matrices defined by channel correlation matrices $\mathbf{R}_{k,i}$ and noise variance $\sigma_{k,i}^2$. When $\mathbf{A}_j$, $\mathbf{B}_j$ are general positive semidefinite matrices, problem (14) serves as a generalization of problem (2). In what follows, we derive SDR approximation bounds for the generalized formulation (14). Clearly, such bounds can be directly applied to problem (2).

Following the same procedure as outlined in Section II, the SDR of problem (14) can be written as

$$v^\star = \max_{\mathbf{W}\in\mathbb{C}^{n\times n}} \min_{j=1,\ldots,M} \frac{\operatorname{tr}(\mathbf{A}_j\mathbf{W})}{\operatorname{tr}(\mathbf{B}_j\mathbf{W})+1} \quad (15a)$$
$$\text{s.t. } \operatorname{tr}(\mathbf{W}) \le P, \quad (15b)$$
$$\mathbf{W} \succeq \mathbf{0}. \quad (15c)$$

The nonconvex fractional quadratic optimization problem (14) can be approximated using the SDR procedure in Table II. We can establish an upper bound on the worst-case approximation ratio $\gamma_{\rm sdr}$ for solving problem (14) by this SDR procedure.

TABLE II
SDR Procedure to Solve Problem (14)

(S1) Solve (15) using the bisection algorithm and let $\mathbf{W}^\star$ be an optimal solution. Let $L > 0$ be an integer.

(S2) For $\ell = 1, 2, \ldots, L$, generate a random vector $\boldsymbol{\xi} \in \mathbb{C}^n$ from the complex Gaussian distribution $\mathcal{N}_c(\mathbf{0}, \mathbf{W}^\star)$, and let

$$\boldsymbol{w}^{(\ell)} = \sqrt{P}\boldsymbol{\xi}/\|\boldsymbol{\xi}\|$$

and

$$u^{(\ell)} = \min_{j=1,\ldots,M} \frac{(\boldsymbol{w}^{(\ell)})^H\mathbf{A}_j\boldsymbol{w}^{(\ell)}}{(\boldsymbol{w}^{(\ell)})^H\mathbf{B}_j\boldsymbol{w}^{(\ell)}+1}. \quad (16)$$

(S3) Suppose

$$\ell^\star = \arg\max_{\ell=1,2,\ldots,L}\{u^{(\ell)}\}.$$

Then select $\boldsymbol{w}^{(\ell^\star)}$ as the approximate optimum solution of (14), and obtain $\hat{u}^\star = u^{(\ell^\star)}$.

Theorem 1: Let $\hat{u}^\star$ and $v^\star$ be obtained by applying the polynomial time SDR procedure to problem (14) and its relaxation problem (15). Then

$$\gamma_{\rm sdr} := \max_{\mathbf{A}_j,\mathbf{B}_j,P,n} \max\left\{\frac{v^\star}{u^\star}, \frac{u^\star}{\hat{u}^\star}\right\} \le 30M \quad (17)$$

holds with probability at least $1-(0.9393)^L$.

Theorem 1 implies that the SDR-based approach has a worst-case $\mathcal{O}(1/M)$ approximation to the optimum solution of $u^\star$. It also implies that the SDR-based approach can have at least an $\mathcal{O}(1/M)$ approximation to the optimum solution of the max-min-fair transmit beamforming problem in (2). The following example demonstrates that this worst-case approximation ratio estimate in (17) is tight up to a constant factor.

Example 1: Consider the special instance of (14) that is originally considered in [14]. Let $\mathbf{A}_j = \boldsymbol{h}_j\boldsymbol{h}_j^H$, where

$$\boldsymbol{h}_j = \left(\cos\left(\frac{\ell\pi}{K}\right), \sin\left(\frac{\ell\pi}{K}\right)e^{i2\pi k/K}, 0,\ldots,0\right)^T \in \mathbb{C}^n$$

in which $i = \sqrt{-1}$, $K = \lceil\sqrt{M}\rceil$, $M \ge 2$ (thus $K \ge 2$), $n \ge 2$ and $j = \ell K - K + k$ with $\ell, k = 1,\ldots,K$. Let $\mathbf{B}_j = \operatorname{diag}\{1/P, 1/P, 0,\ldots,0\} \in \mathbb{R}^{n\times n}$. Without loss of generality, assume that $\boldsymbol{w}^\star = (\rho\cos\theta, \rho(\sin\theta)e^{i\phi},\ldots)^T \in \mathbb{C}^n$ where $\rho > 0$ and $\theta, \phi \in [0, 2\pi)$. It has been shown in [14] that there exists an index $j \in \{1,\ldots,M\}$ such that

$$\left|\boldsymbol{h}_j^H\boldsymbol{w}^\star\right|^2 \le \rho^2\frac{\pi^2(3K+\pi)^2}{4K^4}. \quad (18)$$

By (14) and (18), one can have

$$u^\star \le \frac{\left|\boldsymbol{h}_j^H\boldsymbol{w}^\star\right|^2}{(\boldsymbol{w}^\star)^H\mathbf{B}_j\boldsymbol{w}^\star+1} \le \left|\boldsymbol{h}_j^H\boldsymbol{w}^\star\right|^2 \le \rho^2\frac{\pi^2(3K+\pi)^2}{4K^4} \le \left(\frac{P}{4}\right)\frac{\pi^2(3K+\pi)^2}{K^4}. \quad (19)$$

A feasible point of (15) can be $\bar{\mathbf{W}} = \operatorname{diag}\{P/2, P/2, 0,\ldots,0\}$ which leads to

$$\min_{j=1,\ldots,M} \frac{\operatorname{tr}(\mathbf{A}_j\bar{\mathbf{W}})}{\operatorname{tr}(\mathbf{B}_j\bar{\mathbf{W}})+1} = \frac{P}{4} \le v^\star. \quad (20)$$

Combining (19) and (20) gives

$$v^\star \geq \frac{K^2}{\pi^2(3+\pi/K)^2}u^\star \geq \frac{M}{\pi^2(3+\pi/2)^2}u^\star \tag{21}$$

which indicates that the upper bound given by (17) can be attained within a constant factor. Equation (21) also serves as a theoretical lower bound for the value of $\gamma_{\rm sdr}$. ■

Interestingly, when the data matrices $\mathbf{A}_j$ and $\mathbf{B}_j$ are real-valued and the beamforming vector $\boldsymbol{w}$ is restricted to a real vector, the corresponding worst-case approximation ratio $\gamma_{\rm sdr}$ deteriorates to $\mathcal{O}(M^2)$. One of the possible applications for this real-valued model may be found in the pulse-based ultra-wide-band system [22]–[24] where the transmitted baseband pulses as well as the received signal model are real valued. For theoretical completeness, we provide in Appendix II an approximation ratio analysis for the real-valued formulations of problems (14) and (15). The obtained approximation bound is given by

$$\gamma_{\rm sdr} \leq 80M^2 \tag{22}$$

with probability at least $1-(0.9521)^L$. The upper bound in (22) can also be shown to be tight (up to a constant scalar) via an example similar to Example 1. See Example 2 in [14] for details. Next, let us present the detailed proof of Theorem 1.

A. *Proof of Theorem 1*

The proof of Theorem 1 consists of a probabilistic analysis of the event that the randomized polynomial time SDR procedure in Table II fails to generate a good approximate solution. We will first establish two lemmas which are needed in the proof of Theorem 1. We first bound the rank of the optimal solution of the SDP relaxation (15).

Lemma 1: There exists an optimum solution $\mathbf{W}^\star$ of problem (15) whose rank is upper bounded by $\sqrt{M}$.

Proof: It is easy to show that problem (15) is equivalent to the following optimization problem:

$$P = \min_{\mathbf{W}\in\mathbb{C}^{n\times n}} \mathrm{tr}(\mathbf{W}) \tag{23a}$$

$$\text{s.t. } \frac{\mathrm{tr}(\mathbf{A}_j\mathbf{W})}{\mathrm{tr}(\mathbf{B}_j\mathbf{W})+1} \geq v^\star,\ j=1,\ldots,M, \tag{23b}$$

$$\mathbf{W} \succeq \mathbf{0}. \tag{23c}$$

Since problem (23) is a complex-valued SDP, it has been shown [25] that there exists an optimum solution with $\mathrm{rank}(\mathbf{W}^\star) \leq \sqrt{M}$. ■

We also need the following key lemma to bound the probability that a random fractional quadratic quantity falls in the small neighborhood of the origin.

Lemma 2: Let $\mathbf{A}\in\mathbb{C}^{n\times n}$ and $\mathbf{B}\in\mathbb{C}^{n\times n}$ be two Hermitian positive semidefinite matrices ($\mathbf{A}\succeq\mathbf{0}$ and $\mathbf{B}\neq\mathbf{0},\ \mathbf{B}\succeq\mathbf{0}$), and $\boldsymbol{\xi}\in\mathbb{C}^n$ be a random vector with complex Gaussian distribution $\mathcal{N}_c(\mathbf{0},\mathbf{W}^\star)$. Then

$$\Pr\left(\frac{\boldsymbol{\xi}^H\mathbf{A}\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}\boldsymbol{\xi}+1} < \eta\frac{\mathrm{E}\{\boldsymbol{\xi}^H\mathbf{A}\boldsymbol{\xi}\}}{\mathrm{E}\{\boldsymbol{\xi}^H\mathbf{B}\boldsymbol{\xi}\}+1}\right) \leq \max\left\{\frac{3\eta}{\alpha-2\eta},\left(\frac{5\eta}{\frac{1-\alpha}{\bar{r}-1}-3\eta}\right)^2\right\} \tag{24}$$

where $\bar{r} = \min\{\mathrm{rank}(\mathbf{A}),\mathrm{rank}(\mathbf{W}^\star)\}$, $0\leq\eta<\min\{(\alpha/2),(1/3)(1-\alpha/\bar{r}-1)\}$ and $0<\alpha<1$. The proof of Lemma 2 is presented in Appendix I. With Lemmas 1 to 2, we can prove Theorem 1.

Proof of theorem 1: By Lemma 1, for $M\leq 3$ there exists a solution of problem (15) with $\mathrm{rank}(\mathbf{W}^\star)=1$. Hence, for $M\leq 3$, $v^\star=u^\star$. This rank-1 solution, which is feasible to (14) and has an objective value $\hat{u}^\star$ equal to $u^\star$, can always be obtained via a matrix decomposition procedure [25]. Therefore, $\gamma_{\rm sdr}=1$ for $M\leq 3$. To obtain (17) for $M>3$, we need prove that

$$\Pr\left(\min_{j=1,\ldots,M}\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1}\geq\eta v^\star,\ \boldsymbol{\xi}^H\boldsymbol{\xi}\leq\mu P\right)>0 \tag{25}$$

for $\eta=1/(16M)$ and $\mu=15/8$, where $\boldsymbol{\xi}\in\mathbb{C}^n$ is a random vector with complex Gaussian distribution $\mathcal{N}_c(\mathbf{0},\mathbf{W}^\star)$. If (25) is true, then there exists a realization of $\boldsymbol{\xi}$ which satisfies $\boldsymbol{\xi}^H\boldsymbol{\xi}\leq(15/8)P$ and

$$\min_{j=1,\ldots,M}\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1}\geq\left(\frac{1}{16M}\right)v^\star.$$

Let $\bar{\boldsymbol{\xi}}=\sqrt{8/15}\boldsymbol{\xi}$ which then is feasible for problem (14) (i.e., $\bar{\boldsymbol{\xi}}^H\bar{\boldsymbol{\xi}}\leq P$) and satisfies

$$\left(\frac{1}{30M}\right)v^\star\leq\min_{j=1,\ldots,M}\frac{\bar{\boldsymbol{\xi}}^H\mathbf{A}_j\bar{\boldsymbol{\xi}}}{\bar{\boldsymbol{\xi}}^H\mathbf{B}_j\bar{\boldsymbol{\xi}}+1}\leq u^\star \tag{26}$$

which is part of (17).

We now prove (25). Note that the left-hand side (LHS) of (25) can be lower bounded as follows:

$$\begin{aligned}
&\Pr\left(\min_{j=1,\ldots,M}\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1}\geq\eta v^\star,\ \boldsymbol{\xi}^H\boldsymbol{\xi}\leq\mu P\right)\\
&\geq 1-\sum_{j=1}^M\Pr\left(\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1}<\eta v^\star\right)-\Pr(\boldsymbol{\xi}^H\boldsymbol{\xi}>\mu P)\\
&\geq 1-\sum_{j=1}^M\Pr\left(\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1}<\eta\frac{\mathrm{tr}(\mathbf{A}_j\mathbf{W}^\star)}{\mathrm{tr}(\mathbf{B}_j\mathbf{W}^\star)+1}\right)\\
&\quad-\Pr\left(\boldsymbol{\xi}^H\boldsymbol{\xi}>\mu\cdot\mathrm{tr}(\mathbf{W}^\star)\right)\quad\text{(by (15))}\\
&=1-\sum_{j=1}^M\Pr\left(\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1}<\eta\frac{\mathrm{E}\{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}\}}{\mathrm{E}\{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}\}+1}\right)\\
&\quad-\Pr\left(\boldsymbol{\xi}^H\boldsymbol{\xi}>\mu\mathrm{E}\{\boldsymbol{\xi}^H\boldsymbol{\xi}\}\right)\quad(\text{since }\boldsymbol{\xi}\sim\mathcal{N}_c(\mathbf{0},\mathbf{W}^\star))\\
&\geq 1-\sum_{j=1}^M\Pr\left(\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1}<\eta\frac{\mathrm{E}\{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}\}}{\mathrm{E}\{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}\}+1}\right)-\frac{1}{\mu}\\
&\quad\text{(by Markov inequality)}\\
&\geq\frac{\mu-1}{\mu}-\sum_{j=1}^M\max\left\{\frac{3\eta}{\alpha-2\eta},\left(\frac{5\eta}{\frac{1-\alpha}{\bar{r}-1}-3\eta}\right)^2\right\}
\end{aligned}\tag{27}$$

where the last step in (27) is due to Lemma 2. Because $\bar{r}\leq\sqrt{M}$ by Lemma 1, by choosing $\alpha=0.4932$ and $\eta=1/(16M)$ one can show that for $M>3$,

$$\frac{3\eta}{\alpha-2\eta}\geq\left(\frac{5\eta}{\frac{1-\alpha}{\bar{r}-1}-3\eta}\right)^2. \tag{28}$$

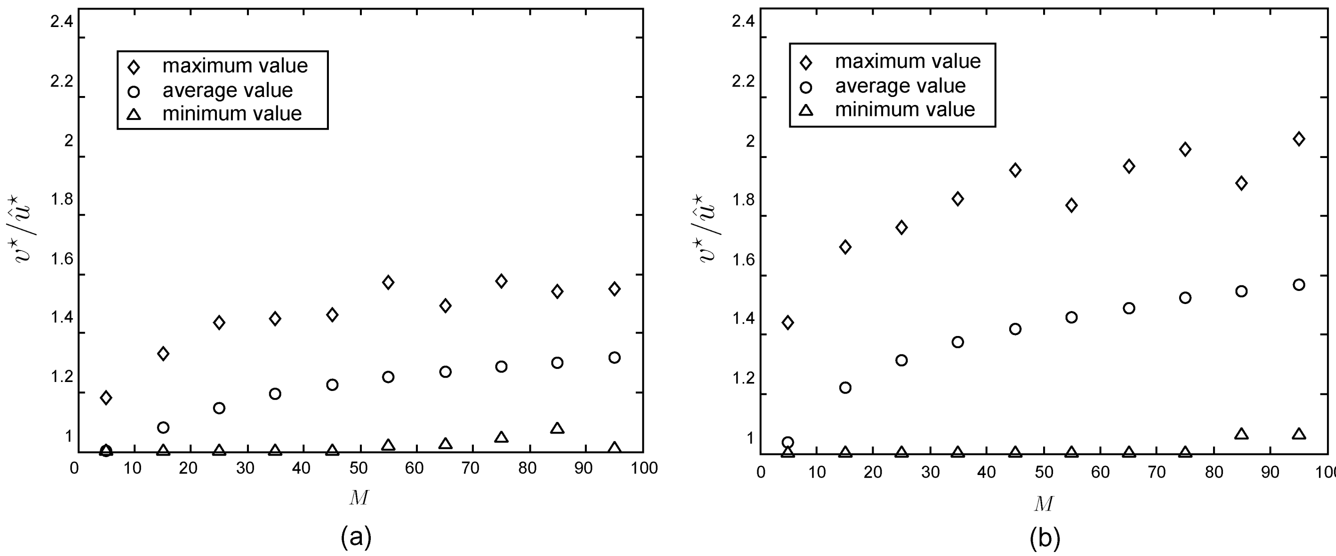

Fig. 1. Empirical approximation ratios for $n = 10$, $P = 10$ and full rank $\mathbf{A}_j$ for (a) complex-valued problem and (b) real-valued problem.

Hence, for $M > 3$ and $\mu = 15/8$,

$$\begin{aligned} \Pr&\left(\min_{j=1,\ldots,M} \frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1} \geq \eta v^\star,\, \boldsymbol{\xi}^H\boldsymbol{\xi} \leq \mu P\right) \\ &\geq \frac{7}{15} - M\frac{3\eta}{\alpha - 2\eta} = \frac{7}{15} - \frac{3}{7.8912 - 2/M} \\ &> 0.0607 \end{aligned} \tag{29}$$

which establishes (25). This further implies that (26) holds.

To complete the proof, let $\hat{u}^*$ be generated by the SDR procedure for solving (14) in Table II. Then, for each ℓ, it follows from (16), (26) and (29) that $\boldsymbol{w}^{(\ell)} = \sqrt{P}\boldsymbol{\xi}/\|\boldsymbol{\xi}\|$ satisfies

$$\begin{aligned} \left(\frac{1}{30M}\right)u^\star &\leq \left(\frac{1}{30M}\right)v^\star \\ &\leq \min_{j=1,\ldots,M} \frac{\left(\boldsymbol{w}^{(\ell)}\right)^H\mathbf{A}_j\boldsymbol{w}^{(\ell)}}{\left(\boldsymbol{w}^{(\ell)}\right)^H\mathbf{B}_j\boldsymbol{w}^{(\ell)}+1} = u^{(\ell)} \end{aligned} \tag{30}$$

with probability at least 0.0607. If one generates L independent realizations of $\boldsymbol{\xi}$ from the distribution $\mathcal{N}_c(\mathbf{0}, \mathbf{W}^\star)$, then it is at least with probability[3] $1 - (1 - 0.0607)^L$ to obtain one $\boldsymbol{\xi}$ which can achieve the approximation quality in (30). Since $\hat{u}^\star = \max\{u^{(1)}, \ldots, u^{(L)}\}$, it follows that

$$\frac{u^\star}{30M} \leq \hat{u}^\star \leq u^\star \tag{31}$$

holds with probability at least $1 - (0.9393)^L$. Theorem 1 is proved. ■

V. Simulations and Discussions

In this section we present simulation results which help illustrate the effectiveness of the approximation bounds in (17) and (22). We generate 1000 random problem instances of (14) and its real-valued counterpart (A19). For each problem instance, the positive semidefinite matrices $\mathbf{A}_j$ and $\mathbf{B}_j$ were generated as follows [26]: For full rank $\mathbf{A}_j$, we set

$$\mathbf{A}_j = \mathbf{Q}^H\mathbf{D}\mathbf{Q} \tag{32}$$

[3]For $L = 50$, this probability is 0.9563.

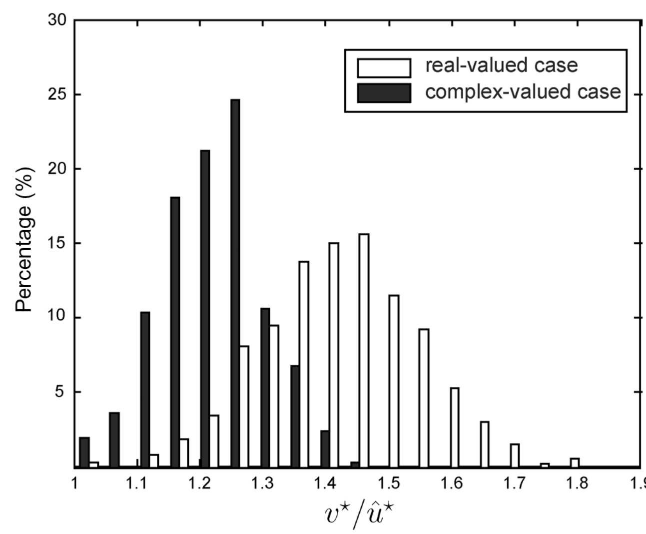

Fig. 2. Histogram of empirical approximation ratios for $n = 10$, $P = 10$, $M = 45$ and full rank $\mathbf{A}_j$.

where $\mathbf{D} = \mathrm{diag}\{\lambda_1, \ldots, , \lambda_n\}$ in which $\lambda_i > 0$ were randomly generated, and $\mathbf{Q} \in \mathbb{C}^{n\times n}$ is a unitary matrix obtained by QR factorization of a randomly generated $n \times n$ complex matrix. For rank-1 $\mathbf{A}_j$, all $\lambda_i = 0$ for $i \neq 1$ while $\lambda_1 > 0$ was randomly generated. Matrices $\mathbf{B}_j$ were generated through the same procedure as full rank $\mathbf{A}_j$. For real-valued $\mathbf{A}_j$ and $\mathbf{B}_j$, we replaced $\mathbf{Q}$ in (32) with an orthonormal matrix obtained by QR factorization of a randomly generated $n \times n$ real-valued matrix. The problems (15) and (A20) [the real-valued counterpart of (15)] were solved by the bisection algorithm [17] wherein SeDuMi [27] was employed to handle the associated semidefinite feasibility problems. The randomized polynomial time SDR procedure in Table II was tested with $L = 50$ randomization candidates generated for each problem instance. Because the empirical approximation ratio $v^\star/\hat{u}^\star$ is greater than or equal to the true ratio $\max\{v^\star/u^\star, u^\star/\hat{u}^\star\}$, the former was used to approximate the latter. For the real-valued problem, the associated SDR procedure is the same as the complex-valued problem, but random vector $\boldsymbol{\xi}$ were drawn from the real Gaussian distribution $\mathcal{N}(\mathbf{0}, \mathbf{W}^\star)$.

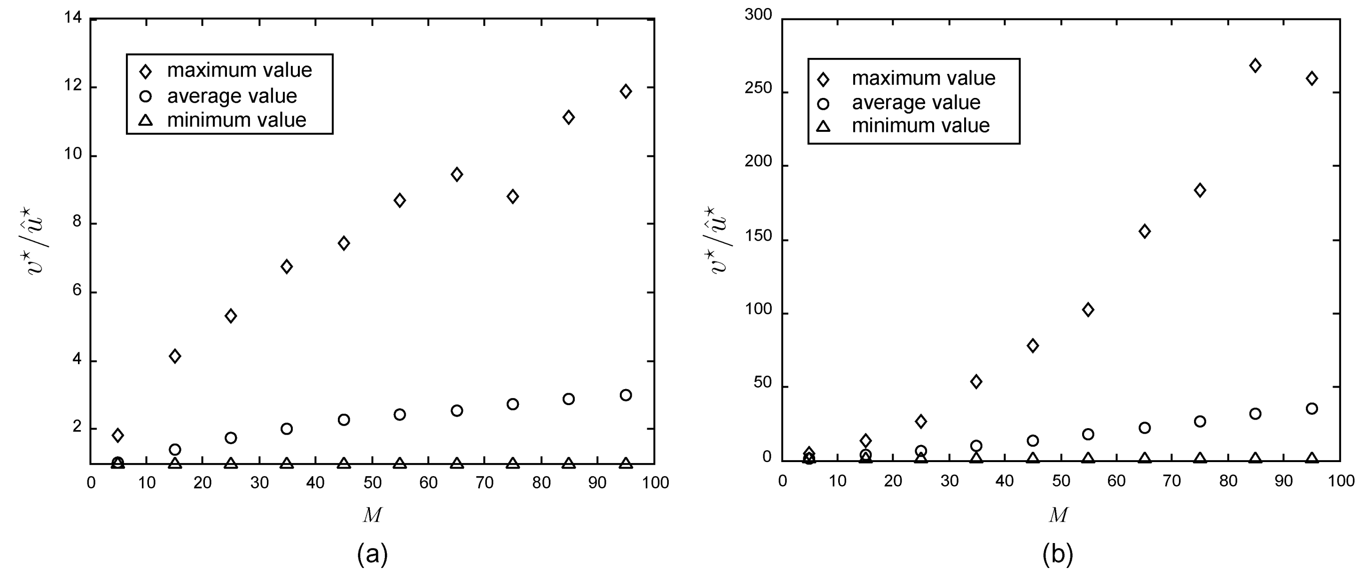

Fig. 3. Empirical approximation ratios for $n=10$, $P=10$ and rank one $\mathbf{A}_j$ for (a) complex-valued problem and (b) real-valued problem.

Simulation Example 1: We first consider the results when matrices $\mathbf{A}_j$ are full rank. Fig. 1 shows the empirical approximation ratios for $n = 10$ and $P = 10$, and Fig. 2 shows the associated histogram for $M = 45$. The symbols "$\Diamond$" ("$\triangle$") denote the maximum (minimum) value of $v^\star/\hat{u}^\star$ for 1000 problem instances, and the symbols "o" represent the average value. One can see from these figures that the empirical approximation ratios get larger when M increases, and the approximation ratios of complex-valued problems are smaller than those of real-valued problems. It can also be seen from these figures that in the average sense the SDR-based approximation method provides very good approximation qualities ($\max\{v^\star/u^\star, u^\star/\hat{u}^\star\} \leq v^\star/\hat{u}^\star < 1.6$ for complex-valued problems).

Simulation Example 2: This example shows the results when matrices $\mathbf{A}_j$ are rank one. Fig. 3 illustrates the empirical approximation ratios for $n=10$ and $P=10$, while Fig. 4 shows the associated histogram for $M=45$. By comparing Figs. 1 and 3, one can observe that the approximation ratios for full rank $\mathbf{A}_j$ are much smaller than those for rank one $\mathbf{A}_j$. One can also observe from Fig. 3 that the maximum values of $v^\star/\hat{u}^\star$ (symbols "$\Diamond$") increase with M roughly in a linear manner for the complex-valued problem [Fig. 3(a)], and in a quadratic manner for the real-valued problem [Fig. 3(b)]. These results coincide with our analytic results and Example 1 in Section IV where the proposed approximation bound in (17) is tight (to the first-order of M) in a specific problem instance with all $\mathbf{A}_j$ being rank one.

Although the presented approximation bound in (17) is tight (up to a constant factor) for general matrices $\mathbf{A}_j$, $\mathbf{B}_j$, in (14), it is possible that this bound is loose when $\mathbf{A}_j$, $\mathbf{B}_j$ have block diagonal matrices as in the case of max-min-fair transmit beamforming problem in (2). We conjecture that the worst-case approximation ratio for the max-min-fair transmit beamforming problem in (2) is $\mathcal{O}(\max\{m_1,\ldots,m_G\})$, instead of $\mathcal{O}(\sum_{k=1}^{G} m_k)$ which is proved in this paper. This conjecture is supported by the simulation results presented in [1] and deserves further analysis in future.

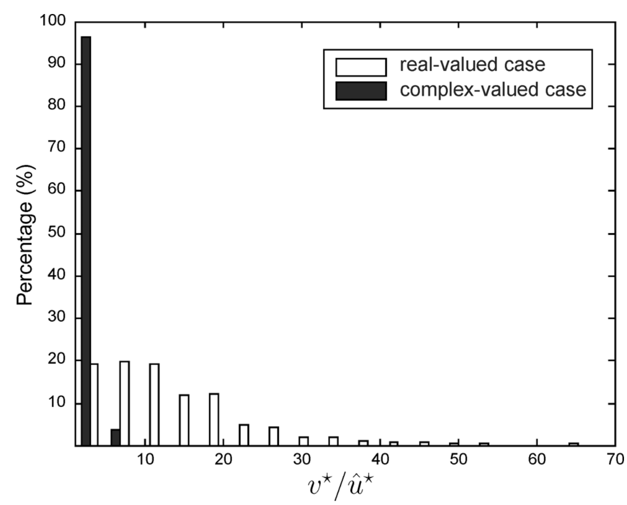

Fig. 4. Histogram of empirical approximation ratios for $n=10$, $P=10$, $M=45$ and rank one $\mathbf{A}_j$.

Appendix I
Proof of Lemma 2

Let $\mathbf{Q} \in \mathbb{C}^n$ be a unitary matrix satisfying

$$\mathbf{Q}(\mathbf{W}^\star)^{1/2}\mathbf{A}(\mathbf{W}^\star)^{1/2}\mathbf{Q}^H = \mathbf{\Lambda}$$

where $\mathbf{\Lambda} = \operatorname{diag}\{\lambda_1,\ldots,\lambda_{\bar{r}},0,\ldots,0\}$ since $\operatorname{rank}(\mathbf{\Lambda}) \leq \bar{r}$, and $\lambda_i \geq 0$. Let $\bar{\boldsymbol{\xi}} = (\mathbf{W}^\star)^{1/2}\mathbf{Q}^H\boldsymbol{x}$, where $\boldsymbol{x} = (x_1,\ldots,x_n)^T \in \mathbb{C}^n$ follows the complex Gaussian distribution $\mathcal{N}_c(0,\mathbf{I}_n)$. Since $\bar{\boldsymbol{\xi}}$ has the same complex Gaussian distribution as $\boldsymbol{\xi}$, we have

$$\begin{aligned}
&\Pr\left(\frac{\boldsymbol{\xi}^H\mathbf{A}\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}\boldsymbol{\xi}+1} < \eta\frac{\mathrm{E}\{\boldsymbol{\xi}^H\mathbf{A}\boldsymbol{\xi}\}}{\mathrm{E}\{\boldsymbol{\xi}^H\mathbf{B}\boldsymbol{\xi}\}+1}\right)\\
&= \Pr\left(\frac{\bar{\boldsymbol{\xi}}^H\mathbf{A}\bar{\boldsymbol{\xi}}}{\bar{\boldsymbol{\xi}}^H\mathbf{B}\bar{\boldsymbol{\xi}}+1} < \eta\frac{\mathrm{E}\{\bar{\boldsymbol{\xi}}^H\mathbf{A}\bar{\boldsymbol{\xi}}\}}{\mathrm{E}\{\bar{\boldsymbol{\xi}}^H\mathbf{B}\bar{\boldsymbol{\xi}}\}+1}\right)\\
&= \Pr\left(\frac{\sum_{i=1}^{\bar{r}}\lambda_i|x_i|^2}{\boldsymbol{x}^H\bar{\mathbf{B}}\boldsymbol{x}+1} < \eta\frac{\sum_{i=1}^{\bar{r}}\lambda_i}{\mathrm{E}\{\boldsymbol{x}^H\bar{\mathbf{B}}\boldsymbol{x}\}+1}\right)\\
&= \Pr\left(\frac{\sum_{i=1}^{\bar{r}}\lambda_i|x_i|^2}{\sum_{i=1}^{\bar{r}}\lambda_i} < \eta\frac{\boldsymbol{x}^H\bar{\mathbf{B}}\boldsymbol{x}+1}{\mathrm{E}\{\boldsymbol{x}^H\bar{\mathbf{B}}\boldsymbol{x}\}+1}\right) \qquad \text{(A1)}
\end{aligned}$$

where $\bar{\mathbf{B}} = \mathbf{Q}(\mathbf{W}^\star)^{1/2}\mathbf{B}(\mathbf{W}^\star)^{1/2}\mathbf{Q}^H \succeq \mathbf{0}$.

Let $\boldsymbol{y} = \mathbf{U}\boldsymbol{x} \in \mathbb{C}^n$ where $\mathbf{U} = [U_{i,j}] \in \mathbb{C}^{n\times n}$ is a unitary matrix satisfying

$$\bar{\mathbf{B}} = \mathbf{U}^H \boldsymbol{\Upsilon} \mathbf{U}$$

where $\boldsymbol{\Upsilon} = \mathrm{diag}\{\mu_1, \ldots, \mu_{\tilde{r}}, 0, \ldots, 0\}$ with $\mu_i \geq 0$ and $\tilde{r} = \min\{\mathrm{rank}(\mathbf{B}), \mathrm{rank}(\mathbf{W}^\star)\}$. Further define

$$\bar{\lambda}_i = \lambda_i / \sum_{i=1}^{\bar{r}} \lambda_i \geq 0, \quad \bar{\mu}_i = \mu_i / \sum_{i=1}^{\tilde{r}} \mu_i \geq 0.$$

Then

$$\sum_{i=1}^{\bar{r}} \bar{\lambda}_i = \sum_{i=1}^{\tilde{r}} \bar{\mu}_i = 1.$$

The right-hand side (RHS) of (A1) can be upper bounded by

$$\begin{aligned}
&\Pr\left(\frac{\sum_{i=1}^{\bar{r}} \lambda_i |x_i|^2}{\sum_{i=1}^{\bar{r}} \lambda_i} < \eta \frac{\boldsymbol{x}^H \bar{\mathbf{B}} \boldsymbol{x} + 1}{\mathrm{E}\{\boldsymbol{x}^H \bar{\mathbf{B}} \boldsymbol{x}\} + 1}\right) \\
&= \Pr\left(\frac{\sum_{i=1}^{\bar{r}} \lambda_i |x_i|^2}{\sum_{i=1}^{\bar{r}} \lambda_i} < \eta \frac{\sum_{i=1}^{\tilde{r}} \mu_i |y_i|^2 + 1}{\sum_{i=1}^{\tilde{r}} \mu_i + 1}\right) \\
&\leq \Pr\left(\sum_{i=1}^{\bar{r}} \bar{\lambda}_i |x_i|^2 < \eta \left(\sum_{i=1}^{\tilde{r}} \bar{\mu}_i |y_i|^2 + 1\right)\right) \qquad \text{(A2)} \\
&\leq \Pr\left(\max_{i=1,\ldots,\bar{r}} \bar{\lambda}_i |x_i|^2 < \eta \left(\sum_{i=1}^{\tilde{r}} \bar{\mu}_i |y_i|^2 + 1\right)\right). \qquad \text{(A3)}
\end{aligned}$$

Without loss of generality, assume that $\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_{\bar{r}} \geq 0$. Consider the case of $\bar{\lambda}_1 > \alpha$ for some $0 < \alpha < 1$. The RHS of (A3) can be further bounded as follows:

$$\begin{aligned}
&\Pr\left(\max_{i=1,\ldots,\bar{r}} \bar{\lambda}_i |x_i|^2 < \eta \left(\sum_{i=1}^{\tilde{r}} \bar{\mu}_i |y_i|^2 + 1\right)\right) \\
&\leq \Pr\left(\alpha |x_1|^2 < \eta \left(\sum_{i=1}^{\tilde{r}} \bar{\mu}_i |y_i|^2 + 1\right)\right) \\
&= \Pr\left(\alpha |x_1|^2 < \eta \left(\sum_{i=1}^{\tilde{r}} \bar{\mu}_i \left|\sum_{j=1}^{n} U_{i,j} x_j\right|^2 + 1\right)\right) \\
&= \Pr\left(\alpha |x_1|^2 < \eta \left(\sum_{i=1}^{\tilde{r}} \bar{\mu}_i \left|U_{i,1} x_1 + \sum_{j=2}^{n} U_{i,j} x_j\right|^2 + 1\right)\right) \\
&\leq \Pr\left(\alpha |x_1|^2 < \eta \left(\sum_{i=1}^{\tilde{r}} \bar{\mu}_i \left(2|U_{i,1}|^2 |x_1|^2 + 2\left|\sum_{j=2}^{n} U_{i,j} x_j\right|^2\right) + 1\right)\right) \qquad \text{(A4)} \\
&\leq \Pr\left(\alpha |x_1|^2 < \eta \left(2|x_1|^2 + 2\sum_{i=1}^{\tilde{r}} \bar{\mu}_i \left|\sum_{j=2}^{n} U_{i,j} x_j\right|^2 + 1\right)\right) \qquad \text{(A5)} \\
&= \Pr\left(|x_1|^2 < \frac{\eta}{\alpha - 2\eta}\left(2\sum_{i=1}^{\tilde{r}} \bar{\mu}_i \left|\sum_{j=2}^{n} U_{i,j} x_j\right|^2 + 1\right)\right) \\
&\quad (\text{since } \alpha > 2\eta) \\
&= \mathrm{E}_{x_j, j=2,\ldots,n}\left\{\Pr\left(|x_1|^2 < \frac{\eta}{\alpha - 2\eta} \times \left(2\sum_{i=1}^{\tilde{r}} \bar{\mu}_i \left|\sum_{j=2}^{n} U_{i,j} x_j\right|^2 + 1\right) \,\Big|\, x_j, j = 2, \ldots, n\right)\right\} \qquad \text{(A6)}
\end{aligned}$$

where (A4) is due to $|\sum_{i=1}^{K} a_i|^2 \leq K \sum_{i=1}^{K} |a_i|^2$, and (A5) is due to $|U_{i,j}|^2 \leq 1$ and $\sum_{i=1}^{\tilde{r}} \bar{\mu}_i = 1$. Because x_i are statistically independent, the probability function in (A6) solely depends on the distribution of x_1. Since x_1 has complex Gaussian distribution $\mathcal{N}_c(0, 1)$, $|x_1|^2$ is exponentially distributed with mean equal to 1. The cumulative distributed function of $|x_1|^2$ can be shown to be upper bounded by

$$F_{|x_1|^2}(y) = 1 - e^{-y} \leq y \qquad \text{(A7)}$$

for $y \geq 0$. Applying (A7) to (A6) gives rise to an upper bound for the LHS of (A1)

$$\begin{aligned}
&\Pr\left(\frac{\boldsymbol{\xi}^H \mathbf{A} \boldsymbol{\xi}}{\boldsymbol{\xi}^H \mathbf{B} \boldsymbol{\xi} + 1} < \eta \frac{\mathrm{E}\{\boldsymbol{\xi}^H \mathbf{A} \boldsymbol{\xi}\}}{\mathrm{E}\{\boldsymbol{\xi}^H \mathbf{B} \boldsymbol{\xi}\} + 1}\right) \\
&\leq \mathrm{E}\left\{\frac{\eta}{\alpha - 2\eta}\left(2\sum_{i=1}^{\tilde{r}} \bar{\mu}_i \left|\sum_{j=2}^{n} U_{i,j} x_j\right|^2 + 1\right)\right\} \\
&= \frac{\eta}{\alpha - 2\eta}\left(2\sum_{i=1}^{\tilde{r}} \bar{\mu}_i \mathrm{E}\left\{\left|\sum_{j=2}^{n} U_{i,j} x_j\right|^2\right\} + 1\right) \\
&= \frac{\eta}{\alpha - 2\eta}\left(2\sum_{i=1}^{\tilde{r}} \bar{\mu}_i \sum_{j=2}^{n} |U_{i,j}|^2 + 1\right) \qquad \text{(A8)} \\
&\leq \frac{3\eta}{\alpha - 2\eta} \qquad \text{(A9)}
\end{aligned}$$

where (A8) is owing to that x_i are zero mean and are statistically independent, and (A9) is due to $\sum_{j=2}^{n} |U_{i,j}|^2 \leq 1$. Thus, we obtain one of the upper bounds in (24).

Consider the case of $\bar{\lambda}_1 \leq \alpha$. Since $(\bar{r}-1)\bar{\lambda}_1 \geq (\bar{r}-1)\bar{\lambda}_2 \geq \bar{\lambda}_2 + \cdots + \bar{\lambda}_{\bar{r}} = 1 - \bar{\lambda}_1 \geq 1 - \alpha$, we have

$$\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \beta \triangleq \frac{1 - \alpha}{\bar{r} - 1}.$$

According to (A2), the LHS of (A1) is upper bounded by

$$\begin{aligned}
&\Pr\left(\sum_{i=1}^{\bar{r}} \bar{\lambda}_i |x_i|^2 < \eta \left(\sum_{i=1}^{\tilde{r}} \bar{\mu}_i |y_i|^2 + 1\right)\right) \\
&\leq \Pr\left(\beta\left(|x_1|^2 + |x_2|^2\right) < \eta \left(\sum_{i=1}^{\tilde{r}} \bar{\mu}_i \left|\sum_{j=1}^{n} U_{i,j} x_j\right|^2 + 1\right)\right)
\end{aligned}$$

$$\leq \Pr\left(\beta\left(|x_1|^2+|x_2|^2\right) < \eta\left(3|x_1|^2+3|x_2|^2+3\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2+1\right)\right) \tag{A10}$$

$$= \Pr\left(|x_1|^2+|x_2|^2 < \frac{\eta}{\beta-3\eta} \times \left(3\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2+1\right)\right)$$

$$(\text{since } \beta > 3\eta)$$

$$\leq \Pr\left(|x_1|^2 < \frac{\eta}{\beta-3\eta}\left(3\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2+1\right), |x_2|^2 < \frac{\eta}{\beta-3\eta}\left(3\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2+1\right)\right), \tag{A11}$$

where (A10) is owing to $|\sum_{i=1}^{K} a_i|^2 \leq K\sum_{i=1}^{K}|a_i|^2$ and $|U_{i,j}|^2 \leq 1$. By using the conditional expectation and from (A11), the LHS of (A1) is further upper bounded by

$$\mathrm{E}_{x_j,j=3,\ldots,n}\left\{\Pr\left(|x_1|^2 < \frac{\eta}{\beta-3\eta} \times \left(3\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2+1\right), |x_2|^2 < \frac{\eta}{\beta-3\eta}\left(3\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2+1\right) \middle| x_j, j=3,\ldots,n\right)\right\} \tag{A12}$$

$$\leq \left(\frac{\eta}{\beta-3\eta}\right)^2 \mathrm{E}\left\{\left(3\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2+1\right)^2\right\} \tag{A13}$$

$$= \left(\frac{\eta}{\beta-3\eta}\right)^2 \mathrm{E}\left\{9\left(\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2\right)^2 + 6\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2+1\right\}$$

$$\leq \left(\frac{\eta}{\beta-3\eta}\right)^2\left(9\mathrm{E}\left\{\left(\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2\right)^2\right\}+7\right) \tag{A14}$$

where we have applied (A7) to the inequality in (A13). The expectation term in (A14) can be expressed as

$$\mathrm{E}\left\{\left(\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2\right)^2\right\} = \mathrm{E}\left\{\sum_{i=1}^{\tilde{r}}\bar{\mu}_i^2\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^4 + \sum_{i\neq k}\bar{\mu}_i\bar{\mu}_k\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2\left|\sum_{j=3}^{n}U_{k,j}x_j\right|^2\right\}$$

$$= \sum_{i=1}^{\tilde{r}}\bar{\mu}_i^2\mathrm{E}\left\{\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^4\right\} + \sum_{i\neq k}\bar{\mu}_i\bar{\mu}_k\mathrm{E}\left\{\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2\left|\sum_{j=3}^{n}U_{k,j}x_j\right|^2\right\}. \tag{A15}$$

Since x_i, $i = 1,\ldots,n$ have complex Gaussian distribution $\mathcal{N}_c(\mathbf{0},\mathbf{I}_n)$, we have

$$\mathrm{E}\{x_i\}=0,\ \mathrm{E}\left\{x_i^2\right\}=0,\ \mathrm{E}\left\{|x_i|^2\right\}=1,\ \mathrm{E}\left\{|x_i|^4\right\}=2,$$
$$\mathrm{E}\{x_ix_j\}=\mathrm{E}\left\{x_ix_j^*\right\}=0,\ \forall\, i\neq j.$$

Then it can be shown that

$$\mathrm{E}\left\{\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2\left|\sum_{j=3}^{n}U_{k,j}x_j\right|^2\right\} = \sum_{j=3}^{n}|U_{i,j}|^2\sum_{j=3}^{n}|U_{k,j}|^2 + \left|\sum_{j=3}^{n}U_{i,j}^*U_{k,j}\right|^2 \leq 2\sum_{j=3}^{n}|U_{i,j}|^2\sum_{j=3}^{n}|U_{k,j}|^2 \tag{A16}$$

where (A16) is due to Cauchy–Schwartz inequality. For $i=k$, (A16) becomes

$$\mathrm{E}\left\{\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^4\right\} \leq 2\left(\sum_{j=3}^{n}|U_{i,j}|^2\right)^2. \tag{A17}$$

Substituting (A16) and (A17) into (A15), one then obtains

$$\mathrm{E}\left\{\left(\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2\right)^2\right\} \leq 2\left(\sum_{i=1}^{\bar{r}}\bar{\mu}_i\sum_{j=3}^{n}|U_{i,j}|^2\right)^2 \leq 2. \tag{A18}$$

By substituting (A18) into (A14), we obtain the other desired bound in (24). ■

APPENDIX II
APPROXIMATION BOUND FOR THE REAL-VALUED PROBLEM

We rewrite problem (14) and its relaxation counterpart (15) with real-valued data as follows:

$$u^\star = \max_{\boldsymbol{w}\in\mathbb{R}^n} \min_{j=1,\ldots,M} \frac{\boldsymbol{w}^T\mathbf{A}_j\boldsymbol{w}}{\boldsymbol{w}^T\mathbf{B}_j\boldsymbol{w}+1} \quad \text{(A19a)}$$

$$\text{s.t. } \boldsymbol{w}^T\boldsymbol{w} \le P, \quad \text{(A19b)}$$

$$v^\star = \max_{\mathbf{W}\in\mathbb{R}^{n\times n}} \min_{j=1,\ldots,M} \frac{\mathrm{tr}(\mathbf{A}_j\mathbf{W})}{\mathrm{tr}(\mathbf{B}_j\mathbf{W})+1} \quad \text{(A20a)}$$

$$\text{s.t. } \mathrm{tr}(\mathbf{W}) \le P, \quad \text{(A20b)}$$

$$\mathbf{W} \succeq \mathbf{0}. \quad \text{(A20c)}$$

The randomized polynomial time SDR procedure for problem (A19) is the same as its complex-valued counterpart (14) in Table II, but the random vectors $\boldsymbol{\xi}$ have real Gaussian distribution $\mathcal{N}(\mathbf{0}, \mathbf{W}^\star)$. The approximation bound for problems (A19) and (A20) is given in the following theorem.

Theorem 2: Let $\hat{u}^\star$ and $v^\star$ be obtained by applying the polynomial time SDR procedure to problem (A19) and its relaxation problem (A20). Then

$$\gamma_{\rm sdr} := \max_{\mathbf{A}_j,\mathbf{B}_j,P,n} \max\left\{\frac{v^\star}{u^\star}, \frac{u^\star}{\hat{u}^\star}\right\} \le 80M^2 \quad \text{(A21)}$$

holds with probability at least $1-(0.9521)^L$.

A. Proof of Theorem 2

To prove Theorem 2, we develop the following two lemmas.

Lemma 3: There exists an optimum solution $\mathbf{W}^\star$ of problem (A20) whose rank is less than $\sqrt{2M}$.

Proof: The proof basically follows the same procedure as the proof for Lemma 1. For the real-valued SDPs, according to [28] there exists an optimum solution with $\mathrm{rank}(\mathbf{W}^\star) < \sqrt{2M}$. ■

Lemma 4: Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$ be two symmetric positive semidefinite matrices ($\mathbf{A} \succeq \mathbf{0}$, and $\mathbf{B} \neq \mathbf{0}$, $\mathbf{B} \succeq \mathbf{0}$), and $\boldsymbol{\xi} \in \mathbb{R}^n$ is a random vector with Gaussian distribution $\mathcal{N}(\mathbf{0}, \mathbf{W}^\star)$. Then

$$\Pr\left(\frac{\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}}{\boldsymbol{\xi}^T\mathbf{B}\boldsymbol{\xi}+1} < \eta\frac{\mathrm{E}\{\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}\}}{\mathrm{E}\{\boldsymbol{\xi}^T\mathbf{B}\boldsymbol{\xi}\}+1}\right) \le \max\left\{\sqrt{\frac{8\eta}{\pi(\alpha-2\eta)}}, \frac{8\eta}{\pi\left(\frac{1-\alpha}{\bar{r}-1}-3\eta\right)}\right\} \quad \text{(A22)}$$

where $\bar{r} = \min\{\mathrm{rank}(\mathbf{A}), \mathrm{rank}(\mathbf{W}^\star)\}$, $0 \le \eta < \min\{(\alpha/2), (1/3)(1-\alpha/\bar{r}-1)\}$ and $0<\alpha<1$.

Proof: We follow the same procedure as the proof for Lemma 2, except that $\mathbf{Q} \in \mathbb{R}^{n\times n}$ and $\mathbf{U} \in \mathbb{R}^{n\times n}$ are orthonormal matrices, and complex conjugate transpose of matrices and vectors reduces to transpose of matrices and vectors. For the case of $\bar{\lambda}_1 > \alpha$, by (A6) we have

$$\begin{aligned}\Pr&\left(\frac{\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}}{\boldsymbol{\xi}^T\mathbf{B}\boldsymbol{\xi}+1} < \eta\frac{\mathrm{E}\{\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}\}}{\mathrm{E}\{\boldsymbol{\xi}^T\mathbf{B}\boldsymbol{\xi}\}+1}\right)\\ &\le \mathrm{E}_{x_j,j=2,\ldots,n}\left\{\Pr\left(|x_1|^2 < \frac{\eta}{\alpha-2\eta} \times\left(2\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=2}^{n}U_{i,j}x_j\right|^2+1\right) \Big| x_j, j=2,\ldots,n\right)\right\}. \end{aligned}\quad \text{(A23)}$$

Because $x_1 \sim \mathcal{N}(0,1)$, $|x_1|^2$ has chi-squared distribution (with degree of freedom equal to one). The cumulative distribution function of $|x_1|^2$ can be shown to be upper bounded by

$$F_{|x_1|^2}(y) = \frac{1}{\sqrt{2\pi}}\int_0^y \frac{1}{\sqrt{u}}\exp\left(-\frac{u}{2}\right)du \le \sqrt{\frac{2}{\pi}y}. \quad \text{(A24)}$$

By applying (A24) to (A23), one can have

$$\begin{aligned}\Pr&\left(\frac{\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}}{\boldsymbol{\xi}^T\mathbf{B}\boldsymbol{\xi}+1} < \eta\frac{\mathrm{E}\{\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}\}}{\mathrm{E}\{\boldsymbol{\xi}^T\mathbf{B}\boldsymbol{\xi}\}+1}\right)\\ &\le \mathrm{E}\left\{\sqrt{\frac{2}{\pi}}\sqrt{\frac{\eta}{\alpha-2\eta}}\sqrt{2\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=2}^{n}U_{i,j}x_j\right|^2+1}\right\}\\ &\le \mathrm{E}\left\{\sqrt{\frac{2}{\pi}}\sqrt{\frac{\eta}{\alpha-2\eta}}\left(\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=2}^{n}U_{i,j}x_j\right|^2+1\right)\right\}\\ &= \sqrt{\frac{2\eta}{\pi(\alpha-2\eta)}}\left(\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\mathrm{E}\left\{\left|\sum_{j=2}^{n}U_{i,j}x_j\right|^2+1\right\}\right)\\ &= \sqrt{\frac{2\eta}{\pi(\alpha-2\eta)}}\left(\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\sum_{j=2}^{n}|U_{i,j}|^2+1\right)\\ &\le \sqrt{\frac{8\eta}{\pi(\alpha-2\eta)}}\end{aligned}\quad \text{(A25)}$$

where (A25) is due to $(a+b)/2 \ge \sqrt{ab}$ for $a,b\ge 0$.

For the case of $\bar{\lambda}_1 \le \alpha$, by (A12) we have

$$\begin{aligned}\Pr&\left(\frac{\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}}{\boldsymbol{\xi}^T\mathbf{B}\boldsymbol{\xi}+1} < \eta\frac{\mathrm{E}\{\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}\}}{\mathrm{E}\{\boldsymbol{\xi}^T\mathbf{B}\boldsymbol{\xi}\}+1}\right)\\ &\le \mathrm{E}_{x_j,j=3,\ldots,n}\left\{\Pr\left(|x_1|^2 < \frac{\eta}{\beta-3\eta}\right.\right.\end{aligned}$$

$$\times\left(3\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2+1\right),|x_2|^2 < \frac{\eta}{\beta-3\eta}\left(3\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2+1\right) \left| x_j, j=3,\ldots,n\right)\Bigg\}. \tag{A26}$$

By applying (A24) to (A26), we then obtain the upper bound

$$\begin{aligned}\Pr&\left(\frac{\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}}{\boldsymbol{\xi}^T\mathbf{B}\boldsymbol{\xi}+1}<\eta\frac{\mathrm{E}\{\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}\}}{\mathrm{E}\{\boldsymbol{\xi}^T\mathbf{B}\boldsymbol{\xi}\}+1}\right)\\ &\le\frac{2\eta}{\pi(\beta-3\eta)}\mathrm{E}\left\{3\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\left|\sum_{j=3}^{n}U_{i,j}x_j\right|^2+1\right\}\\ &=\frac{2\eta}{\pi(\beta-3\eta)}\left(3\sum_{i=1}^{\tilde{r}}\bar{\mu}_i\sum_{j=3}^{n}|U_{i,j}|^2+1\right)\\ &\le\frac{8\eta}{\pi(\beta-3\eta)}.\end{aligned}$$

Then the proof is complete. ■

Proof of theorem 2: From Lemma 3, for $M\le 2$ there exists an optimum solution with rank$(\mathbf{W}^\star)=1$. Hence, $v^\star=u^\star$ for $M\le 2$. By applying a matrix decomposition procedure [29] if necessary, this rank-1 solution can be obtained with the objective value $\hat{u}^\star$ equal to $u^\star$. Therefore, $\gamma_{\rm sdr}=1$ for $M\le 2$. For $M>2$, the approximation bound in (A21) can be proved in the same manner as that in Theorem 1 where (25) must be proved for some data independent η and μ. By (27) and by Lemma 4, we then obtain

$$\Pr\left(\min_{j=1,\ldots,M}\frac{\boldsymbol{\xi}^T\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^T\mathbf{B}_j\boldsymbol{\xi}+1}\ge\eta v^\star,\boldsymbol{\xi}^T\boldsymbol{\xi}\le\mu P\right)\ge\frac{\mu-1}{\mu} -\sum_{j=1}^{M}\max\left\{\sqrt{\frac{8\eta}{\pi(\alpha-2\eta)}},\frac{8\eta}{\pi\left(\frac{1-\alpha}{\bar{r}-1}-3\eta\right)}\right\}.$$

Since $\bar{r}<\sqrt{2M}$ by Lemma 3, by choosing $\alpha=0.2694$ and $\eta=1/(20M^2)$, one can show that for $M\ge 3$,

$$\sqrt{\frac{8\eta}{\pi(\alpha-2\eta)}}\ge\frac{8\eta}{\pi\left(\frac{1-\alpha}{\bar{r}-1}-3\eta\right)}.$$

Hence for $M\ge 3$ and $\mu=4$,

$$\begin{aligned}\Pr&\left(\min_{j=1,\ldots,M}\frac{\boldsymbol{\xi}^T\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^T\mathbf{B}_j\boldsymbol{\xi}+1}\ge\eta v^\star,\boldsymbol{\xi}^T\boldsymbol{\xi}\le\mu P\right)\\ &\ge\frac{3}{4}-M\sqrt{\frac{8\eta}{\pi(\alpha-2\eta)}}\\ &=\frac{3}{4}-\frac{2\sqrt{2}}{\sqrt{\pi(5.388-2/M^2)}}\ge 0.0479.\end{aligned}$$

Thus we have completed the proof of this theorem. ■

References

[1] E. Karipidis, N. D. Sidiropoulos, and Z.-Q. Luo, "Quality of service and max-min-fair transmit beamforming to multiple co-channel multicast groups," *IIEEE Trans. Signal Process.*, vol. 56, no. 3, pp. 1268–1279, Mar. 2008.

[2] M. J. Lopez, "Multiplexing, scheduling, and multicasting strategies for antenna arrays in wireless networks," Ph.D. dissertation, Elect. Eng. and Comp. Sci. Dept., MIT, Cambridge, MA, 2002.

[3] N. D. Sidiropoulos and T. N. Davidson, "Broadcasting with channel state information," in *Proc. IEEE SAM*, Sitges, Spain, Jul. 18–21, 2004, pp. 489–493.

[4] J. Zhang, A. M. Sayeed, and B. D. V. Veen, "Optimal space-time transceiver design for selective wireless broadcast with channel state information," *IEEE Trans. Wireless Commun.*, vol. 3, no. 6, pp. 2040–2050, Nov. 2004.

[5] E. Karipidis, N. D. Sidiropoulos, and Z.-Q. Luo, "Far-field multicast beamforming for uniform linear antenna arrays," *IEEE Trans. Signal Process.*, vol. 55, no. 10, pp. 4916–4927, Oct. 2007.

[6] Y. Gao and M. Schubert, "Group-oriented beamforming for multi-stream multicasting based on quality-of-service requirements," in *Proc. IEEE CAMSAP*, Puerto Vallarta, Mexico, Dec. 13–15, 2005, pp. 193–196.

[7] E. Karipidis, N. D. Sidiropoulos, and Z.-Q. Luo, "Transmit beamforming to multiple co-channel multicast groups," in *Proc. IEEE CAMSAP*, Puerto Vallarta, Mexico, Dec. 13–15, 2005, pp. 109–112.

[8] N. D. Sidiropoulos, T. D. Davidson, and Z.-Q. Luo, "Transmit beamforming for physical-layer multicasting," *IEEE Trans. Signal Process.*, vol. 54, no. 6, pp. 2239–2251, Jun. 2006.

[9] D. P. Palomar, J. M. Cioffi, and M. A. Lagunas, "Joint Tx-Rx beamforming design for multicarrier MIMO channels: A unified framework for convex optimization," *IEEE Trans. Signal Process.*, vol. 51, no. 9, pp. 2381–2401, Sep. 2003.

[10] M. Schubert and H. Boche, "Solution of the multiuser downlink beamforming problem with individual SINR constraints," *IEEE Trans. Veh. Technol.*, vol. 53, no. 1, pp. 18–28, Jan. 2004.

[11] M. X. Goemans and D. P. Williamson, "Improved approximation algorithms for maximum cut and satisfiability problem using semi-definite programming," *J. Assoc. Comput. Mach.*, vol. 42, pp. 1115–1145, 1995.

[12] W.-K. Ma, T. N. Davidson, K. M. Wong, Z.-Q. Luo, and P.-C. Ching, "Quasi-maximum-likelihood multiuser detection using semidefinite relaxation with application to synchronous CDMA," *IEEE Trans. Signal Process.*, vol. 50, no. 4, pp. 912–922, Apr. 2002.

[13] N. D. Sidiropoulos and Z.-Q. Luo, "A semidefinite relaxation approach to MIMO detection for high-order QAM constellations," *IEEE Signal Process. Lett.*, vol. 13, no. 9, pp. 525–528, Sep. 2006.

[14] Z.-Q. Luo, N. D. Sidiropoulos, P. Tseng, and S. Zhang, "Approximation bounds for quadratic optimization with homogeneous quadratic constraints," *SIAM J. Optim.*, vol. 18, no. 1, pp. 1–28, Feb. 2007.

[15] E. Matskani, N. D. Sidiropoulos, Z.-Q. Luo, and L. Tassiulas, "Convex approximation techniques for joint multiuser downlink beamforming and admission control," *IEEE Trans. Wireless Commun.*, to be published.

[16] M. R. Garey and D. S. Johnson*, Computers and Intractability: A Guide to the Theory of NP-Completeness*. San Francisco, CA: Freeman, 1979.

[17] S. Boyd and L. Vandenberghe*, Convex Optimization*. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[18] M. Bengtsson and B. Ottersten, "Optimal and suboptimal transmit beamforming," in *Handbook of Antennas in Wireless Communications*, L. C. Godara, Ed. Boca Raton, FL: CRC Press, Aug. 2001, ch. 18.

[19] F. Rashid-Farrokhi and L. Tassiulas, "Power control and space-time diversity for CDMA systems," in *Proc. IEEE GLOBECOM*, Sydney, Australia, Nov. 8–12, 1998, vol. 4, pp. 2134–2140.

[20] E. Visotsky and U. Madhow, "Optimum beamforming using transmit antenna arrays," in *Proc. IEEE Vehicular Technology Conf.*, Houston, TX, May 16–20, 1999, vol. 1, pp. 851–856.

[21] F. Rashid-Farrokhi, K. J. R. Liu, and L. Tassiulas, "Transmit beamforming and power control for cellular wireless systems," *IEEE J. Sel. Areas Commun.*, vol. 16, no. 8, pp. 1437–1450, Oct. 1998.

[22] K. Usuda, H. Zhang, and M. Nakagawa, "Pre-rake performance for pulse based UWB system in a standardized UWB short-range channel," in *Proc. IEEE WCNC*, Atlanta, GA, Mar. 22–25, 2004, pp. 920–925.

[23] H. Liu, R. C. Qiu, and Z. Tian, "Error performance of pulse-based ultra-wideband MIMO systems over indoor wireless channels," *IEEE Trans. Wireless Commun.*, vol. 4, no. 6, pp. 2939–2944, Nov. 2005.

[24] A. Sibille and V. P. Tran, "Spatial multiplexing in pulse based ultrawideband communications," *Eur. Trans. Telecommun.*, vol. 18, pp. 627–637, May 2007.

[25] Y. Huang and S. Zhang, "Complex matrix decomposition and quadratic programming," of Syst. Eng. and Eng. Manag. Dept., The Chinese University of Hong Kong, Hong Kong, Tech. Rep. SEEM2005-02, 2005.

[26] S. He, Z.-Q. Luo, J. Nie, and S. Zhang, "Semidefinite relaxation bounds for indefinite homogeneous quadratic optimization," *SIAM J. Optim.*, submitted for publication.

[27] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optim. Methods Softw.* vol. 11–12, pp. 625–653, 1999 [Online]. Available: http://sedumi.mcmaster.ca/

[28] G. Pataki, "On the rank of extreme matices in semidefinite programs and the multiplicity of optimal eigenvalues," *Math. Oper. Res.*, vol. 23, pp. 339–358, 1998.

[29] J. F. Sturm and S. Zhang, "On cones of nonnegative quadratic functions," *Math. Oper. Res.*, vol. 28, pp. 246–267, 2003.

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