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Corrections to "An Analysis of Instantaneous Frequency Representation Using Time-Frequency Distributions—Generalized Wigner Distribution"

LJ. Stanković and S. Stanković

In the above paper,[1] due to a problem with telecommunications during the time of production of the above-named paper, we were unable to review the page proofs. Some errors remained or occurred in typesetting. Our corrections follow.

The last equation on page 550 (first column) is split in the wrong place. It should read

$$L\,\phi\,(t+\tau/2L) - L\,\phi\,(t-\tau/2L) = \phi'\,(t)\tau + \frac{1}{L^2}\,\frac{2\phi^{(3)}(t)}{3!}\left(\frac{\tau}{2}\right)^3\cdots;$$

$$\phi(t+\tau/2) - \phi(t-\tau/2) = \phi'\,(t)\tau + \frac{2\phi^{(3)}(t)}{3!}\left(\frac{\tau}{2}\right)^3\cdots.$$

Factor 2π is missing on the right-hand side of (3). It should read

$$ITFT(\omega,t) = 2\pi\delta(\omega - \phi'(t)).$$

In addition, factors $\frac{1}{(2\pi)}$ are missing on the right-hand sides of (5), (9), (10), and (13).

Manuscript received September 15, 1995.

The authors are with Elektrotehnicki facultet, Montenegro, Yugoslavia.

Publisher Item Identifier S 1053-587X(96)03073-5.

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New Cumulant-Based Inverse Filter Criteria for Deconvolution of Nonminimum Phase Systems

Chii-Horng Chen, Chong-Yung Chi, and Wu-Ton Chen

Abstract—**This work proposes a new family of cumulant-based inverse filter criteria $J_{M,m}$, which require a single slice of Mth-order ($M \geq 3$) cumulants, a $(2m)$th-order cumulant, and a $(2M-2m)$th-order cumulant of the inverse filter output where $1 \leq m \leq M-1$, for deconvolution of linear time-invariant (LTI) nonminimum phase systems with only non-Gaussian output measurements contaminated by Gaussian noise. Some simulation results are then presented for a performance comparison of the proposed criteria, Tugnait's criteria, and Chi and Kung's criteria. Finally, conclusions are presented.**

I. INTRODUCTION

In blind deconvolution, we are presented with the problem of restoring the input signal of an unknown linear time-invariant (LTI) signal-distorting system with only noisy output measurements. A widely used approach to this problem is the well-known correlation (second-order statistics) based predictive deconvolution [1], [2] that involves the design of a minimum-phase linear prediction error (LPE) filter used as an inverse filter. However, besides the fact that LPE filters are sensitive to additive noise, when the unknown system is nonminimum phase, allpass distortion (phase distortion) will remain in the deconvolved signal. On the other hand, higher order statistics-based inverse filter criteria have been reported for identification and deconvolution of nonminimum phase LTI systems. Recently, Chi and Wu [3], [4] proposed a unified class of inverse filter criteria using two cumulants, in which Wiggins' criterion [5], Shalvi and Weinstein's criterion [6], and Tugnait's criteria [7] are included in addition to a number of new criteria. Chi and Kung [8] also proposed inverse filter criteria that use all Mth-order ($M \geq 3$) cumulants and thus require much larger computational load than members of Chi and Wu's unified class of inverse filter criteria.

This work proposes a new family of cumulant-based inverse filter criteria that use a single slice of cumulants and two single cumulants (with different orders) of inverse filter output. Section II briefly reviews Chi and Kung's criteria followed by Chi and Wu's unified class of inverse filter criteria. The new family of cumulant-based inverse filter criteria is presented in Section III. Some simulation results are then presented in Section IV. Finally, we draw some conclusions.

II. A BRIEF REVIEW OF CUMULANT-BASED INVERSE FILTER CRITERIA

Assume that data $x(k), k = 0, 1, \cdots, N-1$ were generated from the following convolutional model

$$x(k) = u(k) * h(k) + n(k) = \sum_{i=-\infty}^{\infty} h(i)u(k-i) + n(k) \quad (1)$$

under the four assumptions:

Manuscript received September 14, 1994; revised October 9, 1995. This work was supported by the National Science Council under Grant NSC83-0404-E-007-037. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Athina Petropulu.

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Publisher Item Identifier S 1053-587X(96)03057-7.

- **A1)** The LTI system $h(k)$ is causal stable and a stable inverse filter $h_I(k)$ of the system exists.
- **A2)** The input $u(k)$ is real, zero-mean, stationary, independent identically distributed (i.i.d.), non-Gaussian with Mth-order cumulant γ_M.
- **A3)** the measurement noise $n(k)$ is zero-mean Gaussian with unknown statistics.
- **A4)** $u(k)$ is statistically independent of $n(k)$.

Let $e(k)$ be the output of a stable filter $v(k)$ with input $x(k)$, i.e.

$$e(k) = x(k) * v(k) = u(k) * g(k) + w(k) \tag{2}$$

where $w(k) = n(k) * v(k)$ is also a Gaussian noise sequence by **A3)**, and

$$g(k) = h(k) * v(k) \tag{3}$$

is also a stable filter by **A1)**. Next, let us briefly review some existing cumulant-based inverse filter criteria that lead to the optimal $v(k) \approx \alpha h_I(k-\tau)$ (i.e., $g(k) \approx \alpha\delta(k-\tau)$) where $\alpha \neq 0$ is an unknown scale factor and τ is an unknown time delay.

A. Chi and Kung's Criteria

Chi and Kung [8] proposed the following cumulant-based inverse filter criteria

$$\tilde{J}_M(v(k)) = \frac{1}{C_{M,e}^2(0,0,\cdots,0)} \cdot \sum_{(k_1,\cdots,k_{M-1})\in R_{M,q}} \cdots \sum C_{M,e}^2(k_1,k_2,\cdots,k_{M-1}) \tag{4}$$

where $M \geq 3$, $R_{M,q}$ denotes the domain of support associated with the Mth-order cumulant function of qth-order non-Gaussian moving average (MA) processes and $C_{M,e}(k_1,\cdots,k_{M-1})$ is the Mth-order cumulant function of $e(k)$ given by [10] as follows:

$$C_{M,e}(k_1,\cdots,k_{M-1}) = \gamma_M \sum_{k=-\infty}^{\infty} g(k)\cdot g(k+k_1)\cdots g(k+k_{M-1}). \tag{5}$$

It was also shown in [8] that the optimum $\hat{v}(k)$ by minimizing $\tilde{J}_M$ leads to the associated $\hat{g}(k) = \alpha\delta(k-\tau)$ when $q=\infty$. However, the required computational load increases exponentially with M because the total number of $(M-1)$-dimensional vectors $(k_1,\cdots,k_{M-1})$ included in $R_{M,q}$ is proportional to q^{M-1}. Because $\tilde{J}_M(v(k))$ is a highly nonlinear function of $v(k)$, iterative nonlinear optimization algorithms are used to find the optimum $v(k)$.

B. Chi and Wu's Criteria

Chi and Wu [3], [4] proposed a unified class of inverse filter criteria using a $(2s)$th-order cumulant and an $(l+s)$th-order cumulant

$$\bar{J}_{l+s,2s}(v(k)) = \frac{|C_{l+s,e}(0,\cdots,0)|^{2s}}{|C_{2s,e}(0,\cdots,0)|^{l+s}} \tag{6}$$

where $l > s \geq 1$. It was shown in [4] that the inverse filter estimate obtained by maximizing $\bar{J}_{l+s,2s}(v(k))$ is a consistent estimate (except for an unknown scale factor and an unknown time delay) of the true inverse filter $h_I(k)$ for the two cases that $s=1$ along with $\mathrm{SNR}=\infty$ and that $s>1$ along with finite SNR. Iterative optimization algorithms are also required to find the optimum $v(k)$ because a closed-form solution for $v(k)$ can not be obtained. Note that $\bar{J}_{3,2}(l=2,s=1), \bar{J}_{4,2}(l=3,s=1)$ and $\bar{J}_{6,4}(l=4,s=2)$ were proposed by Tugnait [7]. Wiggins' inverse filter criterion [5] $J_W = E[e^4(k)]/(E[e^2(k)])^2$ is related to $\bar{J}_{4,2}$ by $\bar{J}_{4,2} = |J_W - 3|^2$ and Shalvi and Weinstein's inverse filter criterion [6] is $J_{\mathrm{SW}} = |C_{4,e}(0,0,0)|$ subject to the constraint $E[e^2(k)] = E[u^2(k)]$. Next, let us present a new family of cumulant-based inverse filter criteria that use a single slice of cumulants and two single cumulants (with different orders) of $e(k)$.

III. New Family of Cumulant-Based Inverse Filter Criteria

The new family of cumulant-based inverse filter criteria is described in the following theorem.

Theorem 1: Let $e(k)$ be the output of a stable LTI filter $v(k)$ with the input $x(k)$ generated from (1) under the previous assumptions **A1)** through **A4)**. Let $\hat{v}(k)$ be the optimum $v(k)$ such that

$$J_{M,m}(v(k)) = \frac{|C_{2m,e}(0,\cdots,0)|\cdot|C_{2M-2m,e}(0,\cdots,0)|}{C_{M,e}^4(0,\cdots,0)} \cdot \sum_{k=-\infty}^{\infty} C_{M,e}^2(0,\cdots,0,k_m=k,\cdots,k_{M-1}=k) \tag{7}$$

is minimum when $v(k)=\hat{v}(k)$, where $M\geq 3, 1\leq m\leq M-1$ but $m\neq M/2$ when $M/2$ is odd. Then

$$J_{M,m}(\hat{v}(k)) = \min\{J_{M,m}(v(k))\} = \frac{|\gamma_{2m}\cdot\gamma_{2M-2m}|}{\gamma_M^2} \tag{8}$$

and the associated $\hat{g}(k)$ (see (3)) satisfies that $\hat{g}(k) = \alpha\delta(k-\tau)$ where $\alpha\neq 0$ is an unknown scale factor and τ is an unknown time delay, for the following two cases: (i) $m\neq 1, M-m\neq 1$; (ii) $\mathrm{SNR}=\infty$.

The proof of this theorem is given in the Appendix. Two remarks regarding the proposed criteria given by (7) are worth mentioning here.

R1) The case (i) $m\neq 1, M-m\neq 1$ indicates that the inverse filter associated with $J_{M,m}$ is identifiable for finite signal-to-noise ratio (SNR) only when $J_{M,m}$ does not involve second-order cumulant $C_{2,e}(0)$ (i.e., variance) of $e(k)$. The case (ii) $\mathrm{SNR}=\infty$ indicates that the inverse filter associated with $J_{M,m}$ is always identifiable for this case.

R2) The proposed criteria $J_{M,m}(v(k))$ given by (7) use a single slice of Mth-order cumulants ($\gamma_M\neq 0$), a $(2m)$th-order cumulant ($\gamma_{2m}\neq 0$) and a $(2M-2m)$th-order cumulant ($\gamma_{2M-2m}\neq 0$) of $e(k)$. Moreover, $J_{M,m}(\beta v(k)) = J_{M,m}(v(k))$ for all $\beta\neq 0$ (by (2), (5) and (7)).

Next, let us present how to find the optimum inverse filter $v(k)$ with finite data $x(0),x(1),\cdots,x(N-1)$. Assume that $v(k)$ is a causal finite impulse response (FIR) filter of order equal to L, i.e.

$$e(k) = \sum_{j=0}^{L} v(j)x(k-j). \tag{9}$$

We try to find the optimum $\underline{v} = (v(0),v(1),\cdots,v(L))^T$ such that

$$\hat{J}_{M,m}(\underline{v}) = \frac{|\hat{C}_{2m,e}(0,\cdots,0)|\cdot|\hat{C}_{2M-2m,e}(0,\cdots,0)|}{\hat{C}_{M,e}^4(0,\cdots,0)} \cdot \sum_{k=-K}^{K} \hat{C}_{M,e}^2(0,\cdots,0,k_m=k,\cdots,k_{M-1}=k). \tag{10}$$

is minimum where $\hat{C}_{M,e}(k_1,\cdots,k_{M-1})$ denotes the biased Mth-order sample cumulant function of $e(k)$ and K is a positive integer. The optimum $\underline{v}$ can be found by an iterative gradient-type optimization algorithm. At the nth iteration, $\underline{\hat{v}}_{n-1}$ is normalized by

$||\hat{\underline{v}}_{n-1}|| = 1$ (see **R2)**) and then updated by

$$\hat{\underline{v}}_n = \hat{\underline{v}}_{n-1} - \rho \frac{\partial \hat{J}_{M,m}(\underline{v})}{\partial \underline{v}}\bigg|_{\underline{v}=\hat{\underline{v}}_{n-1}} \tag{11}$$

where ρ is a positive constant. A correlation-based LPE filter can be used to initialize the above algorithm because it has the same amplitude spectrum with the optimum inverse filter except for a scale factor when $\mathrm{SNR} = \infty$. Let us conclude this section with the following two remarks.

R3) The proposed criteria $\hat{J}_{M,m}$, which use $2K+3$ cumulants except the case when $m = M/2$ and m is even ($2K+1$ cumulants $C_{M,e}(0,\cdots,0,k_{M/2}=k,\cdots,k_{M-1}=k)$ for $k=-K,\cdots,K$ are used for this case), are computationally faster than Chi and Kung's criteria $\tilde{J}_M$ given by (4) since the number of cumulants used by the latter is proportional to q^{M-1}, but the former require larger computational load than members of Chi and Wu's unified class of inverse filter criteria (see (6)) using two cumulants.

R4) The parameter L can be chosen large enough such that the optimum $\hat{v}(k) \approx 0$ for k close to zero and k close to L. On the other hand, the parameter K plays the same role as the parameter q in Chi and Kung's criteria (see (4)). In other words, the value chosen for K is a tradeoff of computational load and performance for the proposed criteria. By our experience, a value between 10 and 20 for K is suggested.

IV. Simulation Results

Two simulation examples are to be presented to support the proposed new cumulant-based inverse filter criteria in this section. For ease of later use, let $1, v_b(1), v_b(2), \cdots, v_b(l)$ denote coefficients of the minimum phase lth-order LPE filter obtained by Burg's algorithm [2].

Example 1—Performance Test: A second-order nonminimum phase ARMA system with transfer function (taken from [4], [8]) given by

$$H(z) = \frac{1 - 2.7z^{-1} + 0.5z^{-2}}{1 + 0.1z^{-1} - 0.12z^{-2}} \tag{12}$$

was convolved with a zero-mean exponentially distributed random sequence $u(k)$ to generate synthetic data $x(k)$ for SNR = 5 dB, 10 dB, and 15 dB (white Gaussian noise). The proposed criterion $\hat{J}_{3,1}$ with K set to 10, Tugnait's criteria $\bar{J}_{3,2}, \bar{J}_{4,2}, \bar{J}_{6,4}$, and Chi and Kung's criteria $\tilde{J}_3$ with $q = 10$ were used to estimate the inverse filter $v(k)$ of order $L = 16$ by the previous gradient-type iterative algorithm with an initial condition of $\hat{\underline{v}}_0 = (0,\cdots,0,1,v_b(1),\cdots,v_b(8))^T$.

Thirty independent estimates $\hat{v}(k)$, denoted by $\hat{v}_i(k), i = 1,2,\cdots,30$, (normalized by $||\hat{\underline{v}}_i|| = 1$) with the unknown time delay between $\hat{v}_i(k)$ and the true inverse filter $h_I(k)$ artificially removed were obtained to calculate the mean square error (MSE) defined as

$$\mathrm{MSE} = \frac{1}{30}\sum_{i=1}^{30}\left\{\sum_{k=-9}^{7} |\alpha h_I(k) - \hat{v}_i(k)|^2\right\} \tag{13}$$

where α is an artificial scale factor chosen such that $\sum_{k=-9}^{7} |\alpha h_I(k)|^2 = 1$ because the true inverse filter $h_I(k) \approx 0$ for $k < -9$ and $k > 7$ for this case. The obtained simulation results for $N = 2048$ and 4096 are shown in Table I. From this table, one can see that $\tilde{J}_3$ performs best with much larger computational load than the other criteria. The proposed criterion $\hat{J}_{3,1}$ performs slightly better than $\bar{J}_{3,2}$ because the former uses more necessary third-order cumulants, which coherently provide information about the inverse filter, while the latter ($\bar{J}_{3,2}$) outperforms $\bar{J}_{4,2}$ because fourth-order sample cumulants have larger variance than third-order sample cumulants [10]. The criterion $\bar{J}_{6,4}$ performs worst partly because sixth-order sample cumulants have much larger variance than sample cumulants used by the other criteria, and partly because $C_{4,e}(0,0,0) = E[e^4(k)] - 3(E[e^2(k)])^2$ (in the denominator of $\bar{J}_{6,4}$) always tends to zero resulting in $\bar{J}_{6,4}$ unbounded (maximization of $\bar{J}_{6,4}$) in some of the thirty realizations.

TABLE I

Simulation Results of Example 1. MSE's of the Estimated Inverse Filter Associated with Criteria $\hat{J}_{3,1}$, $\bar{J}_{3,2}$, $\bar{J}_{4,2}$, $\bar{J}_{6,4}$ and $\tilde{J}_3$ for SNR = 5 Db, 10 Db and 15 Db Are Shown in the Table for N = 2048 and 4096, Respectively

N	SNR	MSE				
		$\hat{\mathbf{J}}_{3,1}$	$\bar{\mathbf{J}}_{3,2}$	$\bar{\mathbf{J}}_{4,2}$	$\bar{\mathbf{J}}_{6,4}$	$\tilde{\mathbf{J}}_3$
2048	15 dB	0.0193	0.0190	0.0430	0.7401	0.0141
	10 dB	0.0434	0.0446	0.0792	0.7498	0.0177
	5 dB	0.1246	0.1348	0.2454	0.5802	0.0387
4096	15 dB	0.0093	0.0127	0.0287	0.8087	0.0071
	10 dB	0.0261	0.0358	0.0550	0.7796	0.0106
	5 dB	0.1338	0.1128	0.1548	0.6584	0.0246

Example 2—Seismic Deconvolution: The driving input $u(k)$ that was assumed to be a Bernoulli–Gaussian sequence (taken from [9]) and a third-order nonminimum phase ARMA system (also taken from [9]) were used to generate a set of synthetic data $x(k)$ shown in Fig. 1(a) for $N = 512$ and $\mathrm{SNR} = 15$ dB (white Gaussian noise). Fig. 1(b) shows the predictive deconvolved signal $e_b(k)$ (dotted line) obtained by processing $x(k)$ with an LPE filter $v_b(k)$ of order $L = 30$. One can see from this figure that each spike is associated with a residual wavelet (allpass distortion) in addition to a scale factor, because only the amplitude response of nonminimum phase source wavelet can be equalized by the minimum phase LPE filter. The inverse filter $v(k)$ was also assumed to be a FIR filter of order $L = 30$. Note that $\gamma_3 = 0$ but $\gamma_4 \neq 0$ for this case. The proposed criterion $\hat{J}_{4,1}$ with $K = 16$, Tugnait's criteria $\bar{J}_{4,2}$ and $\bar{J}_{6,4}$ were used to estimate $\underline{v}$ by the previous gradient-type iterative algorithm with an initial condition of $\hat{\underline{v}}_0 = (0,\cdots,0,1,v_b(1),\cdots,v_b(15))^T$. The deconvolved signals $e(k)$ (dotted lines) associated with $\hat{J}_{4,1}, \bar{J}_{4,2}$ and $\bar{J}_{6,4}$ are shown in Figs. 1(c), (d), and (e), respectively, where unknown time delays between $e(k)$ and $u(k)$ were artificially removed. One can see that the deconvolved signals shown in Fig. 1(c) and (d) approximate $u(k)$ (solid lines) well except for a scale factor (without phase distortion). On the other hand, the deconvolved signal $e_b(k)$ shown in Fig. 1(b) offers more information about the true input $u(k)$ (solid line) than the deconvolved signal $e(k)$ (dotted line) shown in Fig. 1(e), which is obviously not a good approximation to $u(k)$. This indicates that higher SNR and larger data length N are required by $\bar{J}_{6,4}$ for acceptable performance.

V. Conclusion

We have presented a new family of cumulant-based inverse filter criteria $J_{M,m}$, described in Theorem 1, for deconvolution of nonminimum phase LTI systems with only non-Gaussian output measurements. The proposed criteria require a single slice of Mth-order ($M \geq 3$) cumulants as well as two other cumulants of order equal to $2m$ and $2M-2m$, respectively (see **R2)**). Two simulation examples were then provided to support the proposed inverse filter criteria $J_{M,m}$. The presented two examples also support that $J_{3,1}$ performs slightly better than Tugnait's criterion $\bar{J}_{3,2}$ and the latter performs better than Tugnait's criterion $\bar{J}_{4,2}$, and that $J_{3,1}$ also performs much better than Tugnait's criterion $\bar{J}_{6,4}$. Chi and Kung's

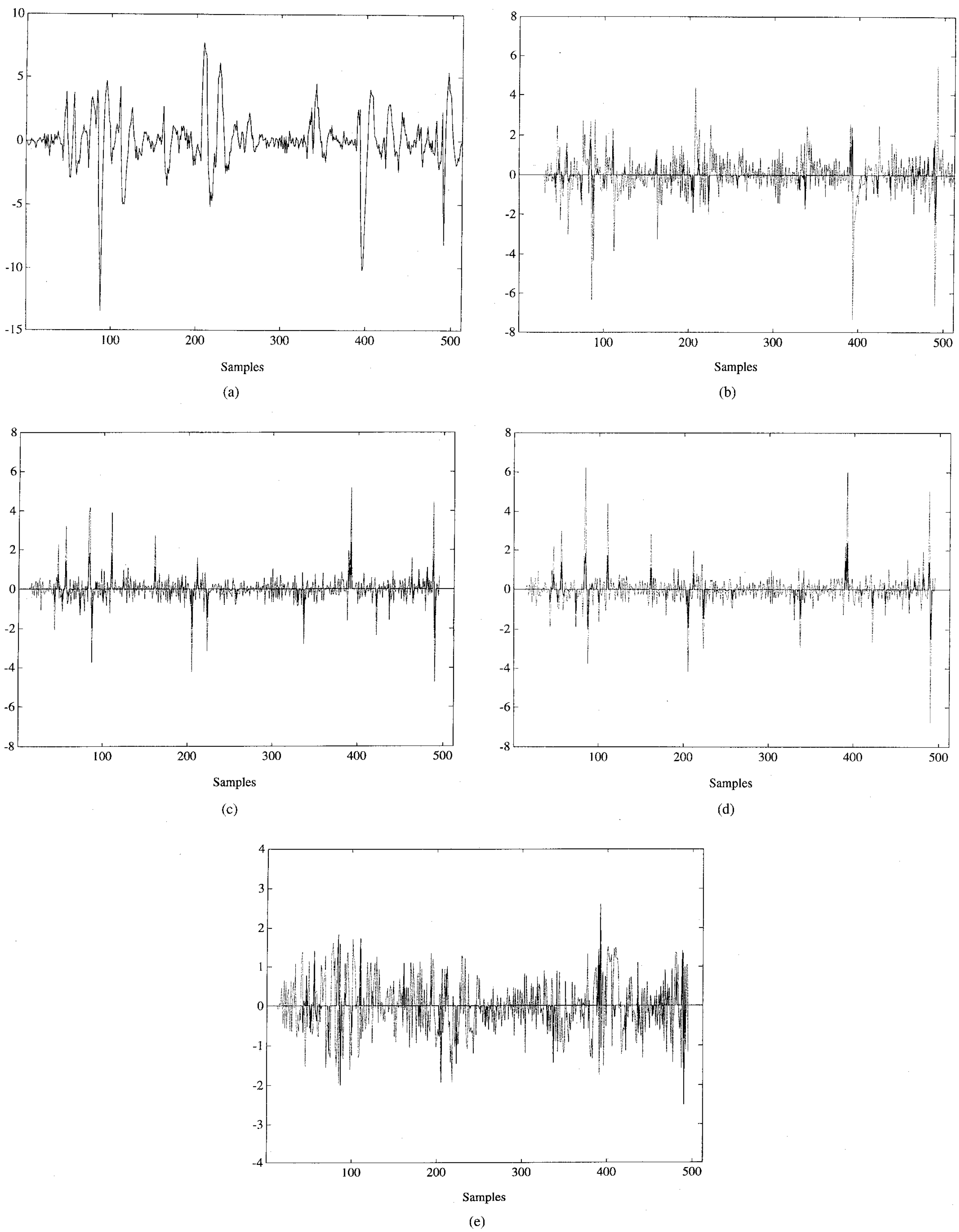

Fig. 1. Simulation results of Example 2: (a) Synthetic noisy data for $N = 512$ and $\mathrm{SNR} = 15$ dB; (b) predictive deconvolved signal $e_b(k)$ (dotted line) together with the true input signal (solid line); (c) deconvolved signal $e(k)$ (dotted line) associated with the proposed criterion $\hat{J}_{4,1}$ together with the true input signal (solid line); (d) deconvolved signal $e(k)$ (dotted line) associated with criterion $\bar{J}_{4,2}$ together with the true input signal (solid line); (e) deconvolved signal $e(k)$ (dotted line) associated with criterion $\bar{J}_{6,4}$ together with the true input signal (solid line).

criteria $\tilde{J}_M$, whose computational load increases exponentially with M, perform better than the proposed criteria $J_{M,m}$, whose computational load is only linearly proportional to the parameter K (see **R3**)), a tradeoff parameter of computational load and performance (see **R4**)). Therefore, when $M \geq 4$, Chi and Kung's criteria $\tilde{J}_M$ may become impractical due to too large computational load.

Appendix
Proof of Theorem 1

The case (i) indicates that all the cumulants used in $J_{M,m}$ have orders greater than two. Moreover, the Mth-order cumulant function of $e(k)$ given by (5) is also true for all $M \geq 2$ for the case (ii). Therefore, (5) can be used in the following proof for these two cases. Substituting (5) into (7) yields

$$\begin{aligned} J_{M,m}(v(k)) &= \frac{|\gamma_{2m} \cdot \gamma_{2M-2m}|}{\gamma_M^2} \\ &\cdot \frac{\{\sum_j g^{2m}(j)\} \cdot \{\sum_j g^{2M-2m}(j)\}}{\{\sum_j g^m(j) \cdot g^{M-m}(j)\}^2} \\ &\cdot \frac{\sum_k \{\sum_j g^m(j) g^{M-m}(j+k)\}^2}{\{\sum_j g^M(j)\}^2} \\ &= \frac{|\gamma_{2m} \cdot \gamma_{2M-2m}|}{\gamma_M^2} \cdot f_1(g(k)) \cdot f_2(g(k)) \quad \text{(A.1)} \end{aligned}$$

where

$$f_1(g(k)) = \frac{\{\sum_j g^{2m}(j)\}\{\sum_j g^{2M-2m}(j)\}}{\{\sum_j g^m(j) \cdot g^{M-m}(j)\}^2} \quad \text{(A.2)}$$

and

$$f_2(g(k)) = \frac{\sum_k \{\sum_j g^m(j) g^{M-m}(j+k)\}^2}{\{\sum_j g^M(j)\}^2}. \quad \text{(A.3)}$$

Note, from (A.1), (A.2), and (A.3), that $J_{M,m}(g(k)) \geq \min\{J_{M,m}(g(k))\} = |\gamma_{2m} \cdot \gamma_{2M-2m}|/\gamma_M^2$ as given by (8) because, for all $g(k) \neq 0$ (zero sequence), $f_1(g(k)) \geq 1$ by the Schwartz inequality and $f_2(g(k)) \geq 1$. The optimum solutions for $g(k)$ associated with $\min\{J_{M,m}(g(k))\}$ must be solutions for both $f_1(g(k)) = 1$ and $f_2(g(k)) = 1$. Hence, let us first find the solution set $S_1 = \{g(k) \mid f_1(g(k)) = 1\}$ and then search S_1 for the $g(k)$ satisfying $f_2(g(k)) = 1$.

By the Schwartz inequality, $f_1(g(k)) = 1$ holds only when

$$g^{M-m}(k) = \beta g^m(k) \quad \text{(A.4)}$$

where $\beta \neq 0$ is a constant. Without loss of generality, let us assume that $\beta > 0$. One can easily infer, from (A.4), that

$$g(k) = \alpha q(k), \quad \text{if } M - 2m \neq 0 \quad \text{(A.5)}$$

where $\alpha = \beta^{1/(M-2m)}$, $q(k)$ is a binary sequence of $\{1, 0\}$ when $M - 2m$ is odd and a trinary sequence of $\{1, 0, -1\}$ when $M - 2m$ is even. Remark that $q(k)$ must be a finite-length sequence, otherwise $g(k)$ is unstable, since $\sum_k |g(k)| = \alpha \sum_k |q(k)| = \infty$ when $M - 2m \neq 0$. On the other hand, when $M = 2m$, $f_1(g(k)) = 1$ (see (A.2)) for all $g(k) \neq 0$. In other words, when $M - 2m \neq 0$, then

$$S_1 = S_{11} = \{g(k) \mid g(k) \neq 0 \text{ is stable and given by (A.5)}\}$$

and when $M = 2m$

$$S_1 = S_{12} = \{g(k) \mid g(k) \neq 0 \text{ is stable}\}.$$

In order to find the optimum $g(k)$ satisfying $f_2(g(k)) = 1$ where $g(k) \in S_1$, let us consider the following three cases: (i) $M - 2m \neq 0$ is odd (M is odd); (ii) $M - 2m \neq 0$ is even (M is even) followed by the case; (iii) $M = 2m$ but m is even.

A. $M - 2m \neq 0$ *Is Odd* (M *Is Odd*): $(g(k) \in S_{11})$

As previously mentioned, $q(k)$ is a finite binary sequence of $\{1, 0\}$ for this case. Let

$$r_{qq}(k) = \sum_i q(i) q(i+k) \quad \text{(A.6)}$$

which is the autocorrelation function of $q(k)$. Substituting (A.5) into (A.3) gives rise to

$$f_2(g(k)) = \frac{1}{r_{qq}^2(0)} \sum_k r_{qq}^2(k) \geq 1 \quad \text{(A.7)}$$

because $q^m(k) = q^{M-m}(k) = q^M(k) = q(k)$. Obviously, the equality in (A.7) holds only when $r_{qq}(k) = 0$ for all $k \neq 0$. In other words, $q(k) = \delta(k - \tau)$ where τ is an arbitrary integer simply because any other binary sequence $q(k) \neq 0$ results in $r_{qq}(k) \neq 0$ for some $k \neq 0$. Therefore, $f_2(g(k)) = 1$ for $g(k) \in S_{11}$ leads to the result $q(k) = \delta(k - \tau)$ or $g(k) = \alpha\delta(k - \tau)$.

B. $M - 2m \neq 0$ *Is Even* (M *Is Even*): $(g(k) \in S_{11})$

Because $q(k)$ is a finite trinary sequences of $\{-1, 1, 0\}$, this case includes two subcases as follows:

m Is Even: Let

$$p(k) = q^2(k) \quad \text{(A.8)}$$

which is also a binary sequence of $\{1, 0\}$. Substituting (A.5) into (A.3) gives

$$f_2(g(k)) = \frac{1}{r_{pp}^2(0)} \sum_k r_{pp}^2(k) \geq 1 \quad \text{(A.9)}$$

since both M and m are even where $r_{pp}(k)$ is the autocorrelation function of $p(k)$. Note that (A.9) is actually the same as (A.7) because both $p(k)$ and $q(k)$ are binary sequences of $\{1, 0\}$. Therefore, as in the previous case, $f_2(g(k)) = 1$ for $g(k) \in S_{11}$, also, leads to the result $p(k) = \delta(k - \tau)$ or $q(k) = \pm\delta(k - \tau)$ or $g(k) = \pm\alpha\delta(k - \tau)$.

m Is Odd: Note that $q^m(k) = q^{M-m}(k) = q(k)$ but $q^M(k) = p(k)$ for this case. Substituting (A.5) into (A.3) gives

$$f_2(g(k)) = \frac{1}{r_{pp}^2(0)} \sum_k r_{qq}^2(k) \geq 1 \quad \text{(A.10)}$$

since $r_{pp}(0) = r_{qq}(0) > 0$, although $r_{pp}(k) \neq r_{qq}(k)$ for $k \neq 0$. Again, the equality in (A.10) holds only when $r_{qq}(k) = 0$ for all $k \neq 0$. What remains to prove is that the trinary sequence $q(k)$ with $r_{qq}(k) = 0$ for $k \neq 0$ is unique and equal to $\pm\delta(k - \tau)$. Assume that $q(\tau) \neq 0$ and that $q(\tau + k) = 0$ for $k < 0$ and for $k \geq l$ since $q(k)$ must be a finite sequence. Then

$$\begin{aligned} r_{qq}(l-1) &= q(\tau) \cdot q(\tau + l - 1) = 0 \Rightarrow q(\tau + l - 1) = 0 \\ r_{qq}(l-2) &= q(\tau) q(\tau + l - 2) + q(\tau + 1) q(\tau + l - 1) \\ &= q(\tau) q(\tau + l - 2) = 0 \Rightarrow q(\tau + l - 2) = 0 \\ &\vdots \\ q(\tau + 1) &= 0. \end{aligned}$$

In other words, $q(k) = \pm\delta(k - \tau)$, or the optimum $g(k)$ associated with $f_2(g(k)) = 1$ for $g(k) \in S_{11}$ too has the form $g(k) = \pm\alpha\delta(k - \tau)$ for this case.

C. $M = 2m$ *but* m *Is Even:* $(g(k) \in S_{12})$

For this case, it can be easily inferred that $f_2(g(k)) = 1$ given by (A.3) for $g(k) \in S_{12}$ leads to

$$g(j) \cdot g(j + k) = 0 \quad \forall k \neq 0 \quad \text{(A.11)}$$

since both m and $M-m$ are even. Assume that $g(\tau) \neq 0$. Then one can easily see, from (A.11), that $g(j) = 0$ for all $j \neq \tau$. In other words, the optimum $g(k)$ has the form $g(k) = \alpha\delta(k-\tau)$. Thus, we have completed the proof.

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Fractal Estimation Using Models on Multiscale Trees

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***Abstract*—In this correspondence, we estimate the Hurst parameter H of fractional Brownian motion (or, by extension, the fractal exponent φ of stochastic processes having $1/f^{\varphi}$-like spectra) by applying a recently introduced multiresolution framework. This framework admits an efficient likelihood function evaluation, allowing us to compute the maximum likelihood estimate of this fractal parameter with relative ease. In addition to yielding results that compare well with other proposed methods, and in contrast with other approaches, our method is directly applicable with, at most, very simple modification in a variety of other contexts including fractal estimation given irregularly sampled data or nonstationary measurement noise and the estimation of fractal parameters for 2-D random fields.**

Manuscript received February 3, 1995; revised November 28, 1995. This work was supported, in part, by the Office of Naval Research under Grant N00014-91-J-1004, the Advanced Research Projects Agency under Grant F49620-93-1-0604, by the Air Force Office of Scientific Research under Grant F49620-95-1-0083, and by an NSERC-67 fellowship of the Natural Sciences and Engineering Research Council of Canada. The associate editor coordinating the review of this paper and approving it for publication was Dr. Petar M. Djuric.

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Publisher Item Identifier S 1053-587X(96)03060-7.

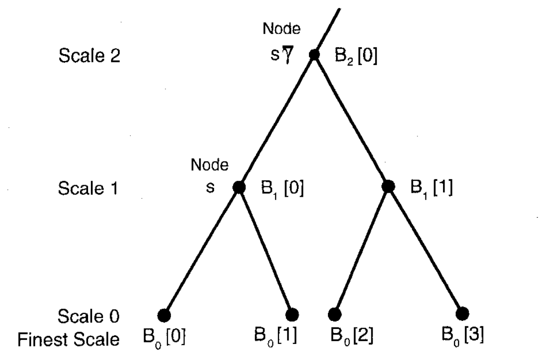

Fig. 1. Dyadic tree structure used for the estimator of this correspondence.

I. INTRODUCTION

Many natural and human phenomena have been found to possess $1/f$-like spectral properties, which has led to considerable study of $1/f$ processes. One class of such processes that is frequently used because of its analytical convenience and tractability is the class of fractional Brownian motion (fBm) processes, which were introduced by Mandelbrot and Van Ness [8]. For practical computation purposes, we consider only sampled versions of continuous time fBm processes $B(t)$, i.e.

$$B[k] = B(k\Delta t) \quad k \in Z \tag{1}$$

for which the associated nonstationary covariance is

$$E\{B[k], B[m]\} = \frac{\sigma^2}{2}(\Delta t)^{2H}(|k|^{2H} + |m|^{2H} - |k-m|^{2H}) \tag{2}$$

where σ and H are scalar parameters that completely characterize the process, and H is the quantity we wish to estimate. Previous estimators have been developed addressing this problem, notably those of Wornell and Oppenheim [11], Kaplan and Kuo [4], Tewfik and Deriche [10], and Flandrin [3]. The exact maximum likelihood (ML) calculation for H is computationally difficult (see [10]); to address this difficulty, fractal estimators typically fall into one of the two following classes to achieve computational efficiency:

1) optimal algorithms, admitting efficient solutions, based on $1/f$-like models other than fBm;
2) approximate or suboptimal algorithms developed directly from the fBm model.

Our approach and that of [11] fall into the former category, whereas the methods in [3], [4], and [9] fall into the latter. In particular, the approach in [11] is based on a $1/f$-like process constructed using wavelets in which the wavelet coefficients are independent, with variances that vary geometrically with scale with exponent H. The method in [4] determines the exact statistics of the Haar wavelet coefficients of the discrete fractional Gaussian noise (DFGN) process $F[k] = B[k+1] - B[k]$ and then develops an estimator by assuming, with some approximation, that the coefficients are uncorrelated.

The goal of our research, on the other hand, is the development of a fast estimator for H that functions under a broader variety of measurement circumstances, for example, the presence of gaps in the measured sequence, measurement noise having a time-varying variance, and higher dimensional processes (e.g., 2-D random fields). The basis for accomplishing this is the utilization of a recently