sequences for $h(n)$, $x(n)$, and $y(n)$, respectively, $n = 0, \cdots, N - 1$. The kernel amplitude spectrum of length $2N$ is divided into $H(2l)$ and $H(2l + 1)$, $l = 0, \cdots, N - 1$, on Fig. 1(d).

We investigated input S/N ratios (denoted by SNR_x) versus output S/N ratios (SNR_y) in four different situations explained by Fig. 2 for the signals of length $N = 256$, and Fig. 3 for lengths $N = 32$. Input SNR's are presented with their average (circles) and the double standard deviation ($\pm\sigma$). The solid lines correspond to deconvolution with interpolated samples, the dashed lines to the direct DFT deconvolution. The (b) parts of the figures are for the singular kernels. We made them singular synthetically by replacing with zeros all the frequency samples $H(k)$ whose magnitudes were lower than 1.5 times the smallest magnitude value.

VI. Conclusion

The proposed method is applicable in the frequency domain when the kernel has no spectral inverse. Its computational complexity is proportional to the complexity of an FFT and it is therefore much faster than equivalent time-domain algorithms [2], [5]. This advantage is obvious in systems performing linear convolution, but is lost in case of periodic convolution because the correction scheme (3.9) must be applied. Section V shows that in a noisy environment, the frequency-domain deconvolution with interpolated samples achieved better input SNR's than the direct DFT algorithm on average.

The method performs better for shorter kernels, as a comparison of Figs. 2 and 3 confirms. Namely, the kernel length N enters the interpolation formula [6], suppressing the distant spectral samples by a factor of N compared to those near the interpolation point. This phenomenon is important when a kernel's energy is concentrated in a very short frequency interval, which causes the interpolated samples distant from this region to be very small. In the deconvolution process, they then excessively magnify the corresponding frequency components in the obtained input signal. That happened in our experiment with four signals with a very high positive offset. By elimination of the kernel's zero-frequency component $H(0)$ contributing to the energy concentration at the beginning of the spectrum a great deal, a remarkable improvement of input SNR's from 19 to 35.29 dB (50 dB at the output) was achieved on average.

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A Phase Determination Method for Nonminimum Phase ARMA Systems by a Single Cumulant Sample

Chong-Yung Chi and Jung-Yuan Kung

***Abstract*—Given a set of output measurements $x(k)$ of a linear time-invariant nonminimum-phase ARMA(p,q) system $H(z)$, the well-known exhaustive search method (ESM) and Giannakis and Mendel's spectrally equivalent minimum-phase (MP) $H_{\mathrm{MP}}(z)$—all-pass (AP) $H_{\mathrm{AP}}(z)$ decomposition based methods include two steps for estimating the coefficients of $H(z) = H_{\mathrm{MP}}(z) \cdot H_{\mathrm{AP}}(z)$. In the first step, $\hat{H}_{\mathrm{MP}}(z)$ is obtained by a correlation-based spectral estimation method. In the second step, the ESM determines $\hat{H}(z)$ from $\hat{H}_{\mathrm{MP}}(z)$ by cumulant matching, and the MP-AP decomposition-based methods identify $\hat{H}_{\mathrm{AP}}(z)$ from the cumulants of the preprocessed data $\hat{u}(k)$, which is the output of the inverse filter $1/\hat{H}_{\mathrm{MP}}(z)$ with input $x(k)$. In this correspondence, we propose a new method only for the second step, which determines the optimum $\hat{H}_{\mathrm{AP}}(z)$ from $\hat{H}_{\mathrm{MP}}(z)$ by maximizing the absolute value of a single Mth-order ($M \geq 3$) cumulant of the output signal of an all-pass filter $H'_{\mathrm{AP}}(z)$ with input $\hat{u}(k)$. The optimum $H'_{\mathrm{AP}}(z) = 1/\hat{H}_{\mathrm{AP}}(z)$. Some simulation results are provided to support the proposed method.**

I. Introduction

Identification of linear time-invariant (LTI) systems with only output measurements is very important in various signal processing application areas such as seismic deconvolution, channel equalization, radar, sonar, biomedical signal processing, radio astronomy, speech processing, and image processing. In the past two decades, a lot of correlation (second-order statistics) based identification methods were reported in the open literature. However, it is well known that the correlation function of measurements is blind to the system phase, and that the system phase can be recovered from higher order statistics of non-Gaussian measurements such as voiced speech signals, binary sequences in digital communications, and seismograms. Recently, cumulant (higher order statistics) based identification [1]–[11] of nonminimum-phase LTI systems with only non-Gaussian output measurements has drawn extensive attention in the previously mentioned signal processing application areas.

Assume that the data $x(k)$, $k = 0, 1, \cdots, N - 1$, were generated from a real stable causal autoregressive moving average (ARMA(p,q)) model without all-pass factors as follows:

$$x(k) = -\sum_{i=1}^{p} a(i)x(k-i) + u(k) + \sum_{i=1}^{q} b(i)u(k-i) \quad (1)$$

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where $u(k)$ is real, zero-mean, independent identically distributed (i.i.d.), non-Gaussian. Equivalently, the system has an irreducible rational transfer function given by

$$H(z) = \frac{B(z)}{A(z)} = \frac{1 + b(1)z^{-1} + \cdots + b(q)z^{-q}}{1 + a(1)z^{-1} + \cdots + a(p)z^{-p}} \tag{2}$$

where the denominator $A(z)$ is minimum phase, and the numerator $B(z)$ can be nonminimum phase. Assume that $B(z)$ has q_1 real roots and q_2 pairs of complex conjugate roots. Then the total number of independent locations of zeros of $B(z)$ is $Q = (q_1 + q_2) \le q = q_1 + 2q_2$. Among the reported cumulant-based system identification methods such as [1]–[11] for nonminimum-phase ARMA(p,q) models, the well-known exhaustive search method (ESM) [1], [2] and Giannakis and Mendel's minimum-phase (MP) all-pass (AP) decomposition based methods [6], [9] basically consist of two steps. The first step includes the estimation of the spectrally equivalent (SE) minimum-phase $H_{\rm MP}(z)$ by a correlation based ARMA spectral estimator. In the second step, the ESM searches for the desired transfer function $\hat{H}(z)$ by minimizing

$$J' = \sum_{k_1}\sum_{k_2}\cdots\sum_{k_{M-1}} |C_M(k_1, k_2, \cdots, k_{M-1}) - \hat{C}_{M,x}(k_1, k_2, \cdots, k_{M-1})|^2 \tag{3}$$

where $\hat{C}_{M,x}(k_1, k_2, \cdots, k_{M-1})$ is the Mth-order sample cumulant of data and $C_M(k_1, k_2, \cdots, k_{M-1})$ is the calculated Mth-order cumulant associated with one of the 2^Q possible SE $\hat{H}(z)$. On the other hand, Giannakis and Mendel's methods [6], [9] (also see [10]), which are based on the following MP-AP decomposition

$$H(z) = H_{\rm MP}(z) \cdot H_{\rm AP}(z) = \frac{B_{\rm MP}(z)}{A(z)} \cdot \frac{B(z)}{B_{\rm MP}(z)} \tag{4}$$

in which $H_{\rm MP}(z) = B_{\rm MP}(z)/A(z)$ is minimum-phase and $H_{\rm AP}(z) = B(z)/B_{\rm MP}(z)$ is an all-pass filter, preprocess the data $x(k)$ with the inverse filter $1/\hat{H}_{\rm MP}(z)$ to obtain a "second-order white" innovations process $\hat{u}(k)$. In [6], $H_{\rm AP}(z)$ is estimated from slices of the sixth-order cumulant function of $\hat{u}(k)$ through a quite complicated procedure. In [9], fourth-order cumulants of $\hat{u}(k)$ are used to estimate the AR parameters of $H_{\rm AP}(z)$ and then the zeros (conjugate reciprocal of poles) of $\hat{H}_{\rm AP}(z)$ are automatically determined.

In this correspondence, we propose a new phase determination method only for the second step of the ESM and Giannakis and Mendel's MP-AP decomposition based methods. In Section II, we present this new phase determination method. Some simulation results are then provided to support the proposed method in Section III. Finally, we draw some conclusions.

II. A New Phase Determination Method

Assume that the order (p,q) of $H(z)$ is known *a priori* and the minimum-phase estimate $\hat{H}_{\rm MP}(z) = \hat{B}_{\rm MP}(z)/\hat{A}(z)$ has been obtained. For ease of later use, let $S_{\rm AP}(z)$ be the set of 2^Q all-pass filters as follows:

$$S_{\rm AP} = \{H'_{\rm AP}(z) | H'_{\rm AP}(z) = \hat{B}_{\rm MP}(z)/\hat{B}(z),\ \hat{B}(z) \in S_B\}$$

where $S_B = \{\hat{B}(z) | \hat{B}(z) \cdot \hat{B}(z^{-1}) = \hat{B}_{\rm MP}(z) \cdot \hat{B}_{\rm MP}(z^{-1})\}$ contains all SE $\hat{B}(z)$ associated with $\hat{B}_{\rm MP}(z)$. Note that $|H'_{\rm AP}(z = \exp\{j2\pi f\})| = 1$ for all $H'_{\rm AP}(z) \in S_{\rm AP}$. The signal processing procedure of the new phase determination method is shown in Fig. 1. Following Giannakis and Mendel's MP-AP decomposition, we also preprocess $x(k)$ by the inverse filter $1/\hat{H}_{\rm MP}(z)$ to obtain the "second-order white" innovations process $\hat{u}(k)$. Furthermore, we process $\hat{u}(k)$ by an all-pass filter $H'_{\rm AP}(z) \in S_{\rm AP}$ such that the Mth-order cumulant sample, $\hat{C}_{M,v}(k_1 = 0, k_2 = 0, \cdots, k_{M-1} = 0)$, of the output signal $v(k)$ of $H'_{\rm AP}(z)$, is maximum in absolute value. The new phase determination method is based on the following theorem.

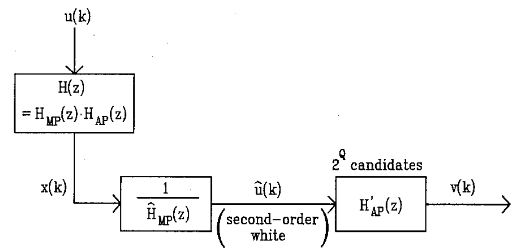

Fig. 1. Signal processing procedure [from $x(k)$ to $v(k)$] of the proposed phase determination method.

Theorem 1: Assume that $x(k)$ is the output of an ARMA(p,q) model given by (1) with input $u(k)$ being zero-mean i.i.d. with nonzero Mth-order $(M \ge 3)$ cumulant γ_M, and that $H_{\rm MP}(z)$ is known *a priori*. Let $x(k)$ be the input of the system shown in Fig. 1. Then the following inequality holds:

$$|C_{M,v}(k_1 = 0, k_2 = 0, \cdots, k_{M-1} = 0)| \le |\gamma_M| \qquad \text{for all } H'_{\rm AP}(z) \in S_{\rm AP}$$

where $C_{M,v}(k_1, k_2, \cdots, k_{M-1})$ denotes the Mth-order cumulant function of the output $v(k)$ of the system. The equality holds if and only if $H'_{\rm AP}(z) = 1/H_{\rm AP}(z)$. The proof of this theorem is given in Appendix A.

Let us summarize the proposed phase determination method as follows:

1) Preprocess $x(k)$ by the inverse filter $1/\hat{H}_{\rm MP}(z)$ to get $\hat{u}(k)$, which is further processed by an all-pass filter $H'_{\rm AP}(z) \in S_{\rm AP}$ (see Fig. 1).
2) Search for the optimum $H'_{\rm AP}(z)$ from $S_{\rm AP}$ such that the objective function
$$J(v(k)) = |\hat{C}_{M,v}(k_1 = 0, k_2 = 0, \cdots, k_{M-1} = 0)| \tag{5}$$
which only involves a single sample cumulant of $v(k)$, is maximum.
3) As a by-product, the estimate $\hat{\gamma}_M$ is determined to be
$$\hat{\gamma}_M = \hat{C}_{M,v}(k_1 = 0, k_2 = 0, \cdots, k_{M-1} = 0) \tag{6}$$
which is associated with $J_{\max} = |\hat{\gamma}_M|$.

Note that every all-pass filter $H'_{\rm AP}(z)$ of $S_{\rm AP}$ except the one $H'_{\rm AP}(z) = 1$ is anticausal stable since all its poles are outside the unit circle. In other words, the all-pass filtering with $H'_{\rm AP}(z)$ must run backwards to obtain the output signal $v(k)$. It is advisable here to indicate the distinctions between the proposed method and some other methods as follows.

R1) Huzzi [8] proposed a method to estimate coefficients of the AR process rather than ARMA process by using a third-order moment of innovations process with lag $(0, -1)$ instead of letting the innovations process pass through an all-pass filter, although he gave an expression for $\hat{\gamma}_3$ in terms of the SE parameters and the cumulant of innovations process with lag $(0, -1)$.

R2) Shalvi asnd Weinstein [11] proposed a blind equalizer for nonminimum phase LTI channels by maximizing the absolute kurtosis (a single fourth-order cumulant) of the output $y(k)$ (corresponding to $v(k)$ in Fig. 1) of equalizer without utilizing the estimate $\hat{H}_{\rm MP}(z)$ under the constraint $E\{|y(k)|^2\} = E\{|c(k)|^2\}$ where

$c(k)$ (corresponding to $u(k)$ in Fig. 1) is the input sequence of channel. The obtained optimum equalizer corresponds to the inverse filter $1/H(z)$ except for a constant delay factor. Their method requires $M = 4$. However, requiring that $\hat{H}_{MP}(z)$ be given in advance, our method maximizes J for any $M \geq 3$ as long as $\gamma_M \neq 0$ under the constraint of all-pass filter with unity gain, which is stronger than the previous Shalvi and Weinstein's constraint since the former implies $E\{|v(k)|^2\} = E\{|u(k)|^2\}$.

Next, we present how to find the J_{max}. Two phase searching algorithms for finding J_{max} are considered in the following.

A. Phase Searching Algorithm 1 (PSA1) (Exhaustive Search)

The first phase searching algorithm, denoted PSA1, simply searches S_{AP} for the desired $H'_{AP}(z)$ by performing $F_1 = 2^Q$ operations of all-pass filtering (see Fig. 1) and then computing the associated F_1 Mth-order sample cumulants $\hat{C}_{M,v}(k_1 = 0, k_2 = 0, \cdots, k_{M-1} = 0)$. In other words, it searches for J_{max} in an exhaustive manner.

B. Phase Searching Algorithm 2 (PSA2) (Iterative Search)

The second phase searching algorithm, denoted PSA2, was motivated by the fact that every all-pass filtering $H'_{AP}(z)$ of S_{AP} can be factored as a cascade of first-order all-pass filters of S_{AP} and second-order all-pass filters of S_{AP}. Note that the overall transfer function from $u(k)$ to $\hat{u}(k)$ corresponds to the all-pass filter $\hat{B}(z)/\hat{B}_{MP}(z)$. The algorithm PSA2 begins with the initial $\hat{B}_0(z)$ and $v(k)$ set to $\hat{B}_{MP}(z)$ and $\hat{u}(k)$, respectively. For the ith iteration, we form the following all-pass filters associated with $\hat{B}_{i-1}(z)$, denoted $H_j(z)$, as follows:

$$H_j(z) = \begin{cases} (z^{-1} - z_j)/(1 - z_j z^{-1}), & \text{for a real } z_j \\ (z^{-1} - z_j^*)(z^{-1} - z_j) \\ (1 - z_j z^{-1})(1 - z_j^* z^{-1}), & \text{for a complex } z_j \end{cases} \quad (7)$$

where $1/z_j$ is the jth root of $\hat{B}_{i-1}(z)$ inside the unit circle. Note that $H_j(z) \in S_{AP}$ for all j. Let $v(k)$ be the common input to all-pass filters $H_j(z)$ for all j. Then we compute $J(v_j(k))$ where $v_j(k)$ is the output of the all-pass filter $H_j(z)$. Then we update J_i by

$$J_i = \max_j \{J(v_j(k))\} = J(v_n(k)). \quad (8)$$

If $J_i > J_{i-1}$, $\hat{B}_i(z)$ and $v(k)$ are updated as follows:

$$\hat{B}_i(z) = \hat{B}_{i-1}(z)/H_n(z) \quad (9)$$

$$v(k) = v_n(k). \quad (10)$$

The overall transfer function from $u(k)$ to $v(k)$ thus corresponds to the all-pass filter $[\hat{B}(z)/\hat{B}_{i-1}(z)] \cdot H_n(z) = \hat{B}(z)/\hat{B}_i(z)$. The signal $v(k)$, therefore, plays the same role as $\hat{u}(k)$ in Fig. 1 for the next iteration. When $J_i < J_{i-1}$, the algorithm converges; $J_{max} = J_{i-1}$, and the overall transfer function from $u(k)$ to $v(k)$ corresponds to the all-pass filter $\hat{B}(z)/\hat{B}_{i-1}(z) = 1$. Thus, $\hat{H}(z) = \hat{B}(z)/\hat{A}(z) = \hat{B}_{i-1}(z)/\hat{A}(z)$. The algorithm PSA2 can be viewed as an iterative discrete search algorithm contrast to PSA1. However, it is not guaranteed that the former can find the global maximum of J. Some characteristics of PSA2 are given as follows:

C1) In each iteration, either a first-order all-pass factor or a second-order all-pass factor of the desired $H'_{AP}(z)$ is determined and the objective function J is guaranteed to increase in the meantime. Therefore, the total number of iterations spent until convergence is equal to OU + 1 where OU is the sum of the number of real roots of $\hat{B}(z)$ outside the unit circle and the number of pairs of complex conjugate roots of $\hat{B}(z)$ outside the unit circle.

C2) Both the total number of all-pass filtering operations performed and the total number of Mth-order sample cumulants computed equal

$$\begin{aligned} F_2 &= Q + (Q - 1) + \cdots + (Q - \text{OU}) \\ &= (\text{OU} + 1)(Q - \tfrac{1}{2}\text{OU}). \end{aligned}$$

It can be easily shown that $F_2 < F_1 = 2^Q$. Therefore, PSA2 is faster than PSA1.

III. Simulation Results

We generated a zero-mean exponentially distributed i.i.d. random sequence $u(k)$ with variance $\sigma^2 = 1$ and skewness $\gamma_3 = 2$. Then we passed this sequence through the selected ARMA model to obtain the noise-free synthetic data $x(k)$. The length of data was $N = 1024$ and cumulant order $M = 3$ was used in our simulation. The improved inverse filtering method [12], [13], which simultaneously estimates ARMA parameters as well as the initial conditions of the inverse filter $1/\hat{H}_{MP}(z)$ from $x(k)$ in the sense of least squares of the output of $1/\hat{H}_{MP}(z)$, was then used to obtain $\hat{H}_{MP}(z) = \hat{B}_{MP}(z)/\hat{A}(z)$. We then preprocessed the data $x(k)$ to obtain $\hat{u}(k)$ and $\hat{w}(k)$ by using the inverse filter $1/\hat{H}_{MP}(z)$ and the filter $\hat{A}(z)$, respectively. Then we searched for $\hat{B}(z)$ by the proposed method (including PSA1 and PSA2) with $\hat{u}(k)$ and by the ESM with $\hat{w}(k)$, which corresponds to a MA (q) process, respectively. Finally, mean and standard deviation for each estimated parameter of $\hat{H}(z)$ and $\hat{\gamma}_3$ were calculated from 30 independent runs.

A. Example 1. Maximum-Phase ARMA(5,4)

The ARMA parameters used were shown in Table I and the zeros of this model are located at 2, 1.4, 0.9 ± 0.9j. For this case $q_1 = 2$, $q_2 = 1$, $Q = q_1 + q_2 = 3$ and OU = 3. The simulation results for ARMA parameters, which are shown in Table I, were exactly the same for both our method and the ESM. As we predicted, the number of iterations spent by PSA2 in each run was equal to OU + 1 = 4 ($<2^Q = 8$ associated with PSA1). Additionally, the mean ± standard deviation for the estimate $\hat{\gamma}_3$ obtained by our method was 1.9703 ± 0.0652 and that obtained by the ESM was 2.0629 ± 0.0374. These results indicate that both our method and the ESM work well and have the same performance.

B. Example 2. Minimum-Phase ARMA(6,5)

The ARMA parameters used were shown in Table II and the zeros of this model are located at 0.8, 0.75, 0.5, 0.5881 ± 0.5882j. For this case $q_1 = 3$, $q_2 = 1$, $Q = q_1 + q_2 = 4$ and OU = 0. Again, the simulation results for ARMA parameters, which are shown in Table II, were exactly the same for both our method and the ESM. Again, as we predicted, the number of iterations spent by PSA2 in each run was equal to OU + 1 = 1 ($\ll 2^Q = 16$ associated with PSA1). Additionally, the mean ± standard deviation for the estimate $\hat{\gamma}_3$ obtained by our method was 1.9874 ± 0.0104, which is slightly better than the corresponding result 2.1365 ± 0.0429 obtained by the ESM. These results also indicate that both our method and the ESM work well and have the same performance. As a final remark, we also performed many other simulations that ended up with the same conclusion.

TABLE I
SIMULATION RESULTS ASSOCIATED WITH EXAMPLE 1 (MAXIMUM-PHASE ARMA(5,4)), $N = 1024$)

Parameters	True Values	Estimated Values (Mean $\pm \sigma$)
$a(1)$	−3.5	−3.4959 ± 0.0287
$a(2)$	5.0775	5.0715 ± 0.0994
$a(3)$	−3.9013	−3.9040 ± 0.1473
$a(4)$	1.6125	1.6199 ± 0.1104
$a(5)$	−0.2869	−0.2896 ± 0.0340
$b(1)$	−5.2	−5.2001 ± 0.0058
$b(2)$	10.54	10.5403 ± 0.0152
$b(3)$	−10.548	−10.5487 ± 0.0164
$b(4)$	4.536	4.5363 ± 0.0073

TABLE II
SIMULATION RESULTS ASSOCIATED WITH EXAMPLE 2 (MIMIMUM-PHASE ARMA(6,5)), $N = 1024$)

Parameters	True Values	Estimated Values (Mean $\pm \sigma$)
$a(1)$	−1.95	−1.9482 ± 0.0410
$a(2)$	0.4475	0.4473 ± 0.0772
$a(3)$	1.0944	1.0855 ± 0.0591
$a(4)$	−0.5492	−0.5374 ± 0.0639
$a(5)$	−0.1145	−0.1171 ± 0.0605
$a(6)$	0.0766	0.0746 ± 0.0340
$b(1)$	−3.2265	−3.2266 ± 0.0002
$b(2)$	4.4788	4.4793 ± 0.0014
$b(3)$	−3.3363	−3.3372 ± 0.0018
$b(4)$	1.3045	1.3054 ± 0.0016
$b(5)$	−0.2076	−0.2079 ± 0.0006

IV. CONCLUSIONS

In this correspondence, we have presented a new phase determination method (see Fig. 1) which also begins with the same preprocessing by the inverse filter $1/H_{MP}(z)$ with Giannakis and Mendel's MP-AP decomposition based methods and then identifies the all-pass filter $H_{AP}(z)$ by maximizing the objective function J given by (5), which only involves a single Mth-order ($M \geq 3$) cumulant of the output signal $v(k)$ of the all-pass system shown in Fig. 1. Two phase searching algorithms, PSA1 and PSA2, were presented to find the maximum of J. The former is an exhaustive search algorithm whereas the latter is an iterative search algorithm. The proposed method can be implemented either by 2^Q processors in parallel (associated with PSA1) or Q processors (associated with PSA2) in parallel while 2^Q processors are required by the ESM. We prefer PSA2 to PSA1 since PSA2 not only has a much simpler parallel processing structure but also is faster than PSA1 [see characteristic (C2) in Section II-B]. The proposed method was developed for a general case of $M \geq 3$. Some simulation results for the case of $M = 3$ with noise-free synthetic data were also provided to support the proposed method. Finally, we would like to emphasize that the proposed method performs as well as the ESM since their performance is determined solely by the estimation accuracy of the estimated $H_{MP}(z)$.

APPENDIX A
PROOF OF THEOREM 1

The Mth-order cumulant function of $u(k)$ is known to be

$$C_{M,u}(k_1, k_2, \cdots, k_{M-1}) = \gamma_M \delta(k_1)\delta(k_2)\delta(k_3) \cdots \delta(k_{M-1}) \tag{A1}$$

where $\delta(k)$ is the discrete delta function. One can see, from Fig. 1, that $v(k)$ is the output of the all-pass filter $\tilde{H}_{AP}(z) = H_{AP}(z) \cdot H'_{AP}(z)$ with real coefficients. Assume that

$$\tilde{H}_{AP}(f) = \tilde{H}_{AP}(z = \exp\{j2\pi f\}) = \exp\{j2\pi\phi(f)\} \tag{A2}$$

with $\phi(0) = 0$ without loss of generality, where the phase $\phi(f)$ is a continuous odd function of f. The Mth-order polyspectrum, $S_{M,v}(f_1, \cdots, f_{M-1})$, of $v(k)$ is then given by

$$\begin{aligned} S_{M,v}(f_1, f_2, \cdots, f_{M-1}) &= S_{M,u}(f_1, f_2, \cdots, f_{M-1}) \left\{ \prod_{i=1}^{M-1} \tilde{H}_{AP}(f_i) \right\} \\ &\quad \cdot \tilde{H}^*_{AP}(f_1 + \cdots + f_{M-1}) \\ &= \gamma_M \cdot \exp\{j2\pi P(f_1, \cdots, f_{M-1})\} \end{aligned} \tag{A3}$$

where

$$\begin{aligned} P(f_1, \cdots, f_{M-1}) = \phi(f_1) + \cdots + \phi(f_{M-1}) \\ - \phi(f_1 + \cdots + f_{M-1}) \end{aligned} \tag{A4}$$

is also a real continuous function of $f_1, \cdots, f_{M-1}$. Then we, from (A3), obtain

$$\begin{aligned} &|C_{M,v}(0, 0, \cdots, 0)| \\ &= \left| \gamma_M \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} \exp\{j2\pi P(f_1, \cdots, f_{M-1})\} \cdot df_1\, df_2 \cdots df_{M-1} \right| \\ &\leq |\gamma_M| \int_{-1/2}^{1/2} \cdots \int_{-1/2}^{1/2} |\exp\{j2\pi P(f_1, \cdots, f_{M-1})\}| \cdot df_1\, df_2 \cdots df_{M-1} = |\gamma_M|. \end{aligned} \tag{A5}$$

It is trivial to see that if $\tilde{H}_{AP}(z) = 1$, the equality in (A5) holds. Next, we show that when the equality in (A5) holds, $\tilde{H}_{AP}(z) = 1$.

From (A4) and (A5), one can infer that when the equality holds

$$\begin{aligned} \phi(f_1) + \cdots + \phi(f_{M-1}) - \phi(f_1 + \cdots + f_{M-1}) \\ = \theta + L \quad \text{for all } (f_1, \cdots, f_{M-1}) \end{aligned} \tag{A6}$$

where $-1/2 < \theta < 1/2$ is a constant and L is an integer. Letting $f_1 = f_2 = \cdots = f_{M-1} = 0$ in (A6), we obtain $\theta + L = (M - 2)\phi(0) = 0$ since $\phi(0) = 0$. This leads to $\theta = -L$, which implies $\theta = L = 0$ since $-1/2 < \theta < 1/2$. Therefore, (A6) reduces to

$$\phi(f_1) + \cdots + \phi(f_{M-1}) = \phi(f_1 + \cdots + f_{M-1}) \tag{A7}$$

which implies $\phi(\cdot)$ is a linear operator, or $\phi(f) = \alpha \cdot f$, or

$$\tilde{H}_{AP}(z) = H_{AP}(z) \cdot H'_{AP}(z) = z^{\alpha} \tag{A8}$$

where α is a constant. Since $H_{AP}(z) = B(z)/B_{MP}(z)$ never takes the form z^{α_1} with $\alpha_1 \neq 0$ and $H'_{AP}(z) = z^{\alpha_2} \notin S_{AP}$ for any $\alpha_2 \neq 0$, (A8) holds only when $\alpha = 0$. Therefore, $\tilde{H}_{AP}(z) = 1$ when the equality in (A5) holds, or equivalently $H'_{AP}(z) = \hat{B}_{MP}(z)/\hat{B}(z) = B_{MP}(z)/B(z) = 1/H_{AP}(z) \in S_{AP}$ since $H_{MP}(z)$ (or equivalently $\hat{B}_{MP}(z) = B_{MP}(z)$ and $\hat{B}(z) = B(z) \in S_B$) was assumed known. We thus have completed the proof. Q.E.D.

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Asymptotic Distribution of the MUSIC Null Spectrum

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***Abstract*—In this correspondence we derive the *asymptotic distribution* of the MUSIC null spectrum, from which an exact expression of the asymptotic variance of the MUSIC null spectrum can be obtained. The result presented in this correspondence is a simpler alternative and a special case of a more general result recently obtained by Lee and Wengrovitz.**

I. Introduction

For source localization purposes array sensors have been used widely and various high-resolution methods based on the eigenstructure have been proposed, e.g., multiple signal classification (MUSIC) [4] and minimum-norm [2] are among the typical techniques. In [1] and [6], the mean and variance of the MUSIC null spectrum have been obtained to the first-order approximation. It is pointed out in [6], however, that the variance of the MUSIC null spectrum cannot be exactly expressed by the first-order approximation. This is quite an important and reasonable observation, since in general the mean is expressed by a first-order approximation and the variance is usually expressed by a second-order approximation. In this correspondence we obtain the *asymptotic distribution* of the MUSIC null spectrum, from which we can immediately find a more exact expression of the variance.

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II. Background

Let us consider an L-element array of what output is $y(t) \in C^{L \times 1}$ with $C^{L \times 1}$ denoting the space of $L \times 1$ complex-valued vectors, and assume the standard model of observation

$$y(t) = Ax(t) + n(t), \qquad 1, 2, \cdots, N. \tag{2.1}$$

In (2.1) it is assumed that the column vector $x(t)$ for M-signal sources is an $M \times 1$ zero mean complex normal random vector and the additive noise $n(t)$ is also a zero mean complex normal random vector with covariance matrix σI. The full-rank covariance matrix of $x(t)$ is $E[x(t)x^H(t)] = R_x$ where E denotes the statistical expectation and H denotes the Hermitian transpose. The matrix A is an $L \times M$ $(L > M)$ complex matrix having the particular structure: $A = [a(\theta_1), a(\theta_2), \cdots, a(\theta_M)]$, where θ_i, is the DOA of the ith signal source. Here $a(\theta_i) \in C^{L \times 1}$ is called the steering or transfer vector. If we denote the covariance matrix of $y(t)$ by R_y, it is easy to see that

$$R_y = AR_xA^H + \sigma I. \tag{2.2}$$

The eigenvalues and eigenvectors of R_y are denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L$ and $e_1, e_2, \cdots, e_L$, respectively. It is noteworthy that $\lambda_{M+1} = \lambda_{M+2} = \cdots = \lambda_L = \sigma$. The ranges of the matrices $S \triangleq [e_1, e_2, \cdots, e_M]$ and $G \triangleq [e_{M+1}, e_{M+2}, \cdots, e_L]$ are called the signal and noise subspaces, respectively. We observe that

$$a^H(\theta)G = 0, \qquad \text{for } \theta \in \Theta \tag{2.3}$$

where $\Theta \triangleq \{\theta_1, \theta_2, \cdots, \theta_M\}$, because the vectors $\{a(\theta_i), 1 \leq i \leq M\}$ are orthogonal to the noise subspace. If we define $f(\theta) = a^H(\theta)GG^Ha(\theta)$, the function $f(\theta)$ has zeros only at $\theta \in \Theta$ [5]. In practice, however, we can obtain only the estimates of S and G, $\hat{S}$ and $\hat{G}$, from the estimate of R_y, $\hat{R}_y = (1/N) \Sigma_{i=1}^N y(t)y^H(t)$. The MUSIC null-spectrum $D(\theta)$ is then defined by

$$D(\theta) = a^H(\theta)\hat{G}\hat{G}^Ha(\theta) \tag{2.4}$$

and it is thus expected that $D(\theta)$ has minimum points at around $\theta \in \Theta$. Therefore, we can estimate the DOA by taking the local minimum points of $D(\theta)$.

III. Asymptotic Distribution of the MUSIC Null Spectrum

To establish the distribution of the MUSIC null spectrum, we first review the statistical properties of eigenvectors of the sample covariance matrix $\hat{R}_y$. Following [5, lemma 3.1], the orthogonal projections of $\{\hat{e}_i\}$, $M \times 1 \leq i \leq L$, onto the column space of the signal subspace S are asymptotically jointly Gaussian distributed