all close to CRB's. The unweighted MD-MUSIC algorithm performs notably worse.

V. Conclusion

In this correspondence, we have quantized the effects of the finite number of snapshots on the DOA estimates derived with subspace fitting methods. We provided, for the first time, bias expressions for the totally weighted subspace fitting method estimator. The analysis is valid for all TWSF methods.

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A New Cumulant Based Parameter Estimation Method for Noncausal Autoregressive Systems

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***Abstract*—This correspondence proposes a new nonlinear parameter estimation method for a noncausal autoregressive (AR) system based on a new quadratic equation relating the unknown AR parameters to higher order ($\geq$3) cumulants of nonGaussian output measurements in the presence of additive Gaussian noise. A gradient-type numerical optimization algorithm is used to search for the optimal AR parameter estimates. It is applicable regardless of whether or not the order of the system is known in advance; it is also applicable for the case of the causal AR system. Some simulation results are offered to justify that the proposed method is effective.**

I. Introduction

Autoregressive (AR) system identification with only output measurements is a well-defined problem in various science and engineering areas such as spectral estimation, speech processing, seismology, sonar, radar, radio astronomy, biomedicine, image processing, vibration analysis, and oceanography. Although most existing AR parameter estimation methods assume that the unknown AR model is causal stable, there are some cases wherein the underlying signal generation model is noncausal, which occurs in astronomical signal processing, image processing, and geophysical signal processing.

A known fact is that correlation-based AR parameter estimation methods such as the existing AR spectral estimators are inherently phase blind and sensitive to additive noise, no matter whether the signal of interest is Gaussian or not. Recently, signal processing with higher order statistics known as cumulants has drawn extensive attention because cumulants can be used to extract not only the amplitude information but also the phase information of nonGaussian signals, and they are totally zero for Gaussian processes.

Various cumulant-based AR parameter estimation methods have been reported in the literature. Most of them such as [1]–[6], [11], [12], [14], [15] are only applicable in the case of causal stable AR models; nevertheless, some cumulant-based approaches have been proposed to identify a noncausal AR model, which is denoted $1/A(z)$. For instance, Tugnait's exhaustive search method [7] and minimum phase-allpass (MP-AP) decomposition-based method [8] and Huzii's method [9] begin with the estimation of the spectrally equivalent (SE) minimum-phase system $\widehat{A}_{\mathrm{MP}}(z)$ by correlation-based AR parameter estimation methods. From the set of all AR models SE to $1/\widehat{A}_{\mathrm{MP}}(z)$, the exhaustive search method determines the noncausal $1/A(z)$ to be the candidate whose output cumulants match the corresponding sample cumulants best. For Huzii's method and the MP-AP decomposition-based method, the given nonGaussian data $x(k)$ are processed by the filter $\widehat{A}_{\mathrm{MP}}(z)$ to obtain an innovations process $\widehat{u}(k)$, and the desired noncausal system $1/A(z)$ is then determined from cumulants of $\widehat{u}(k)$. For instance, Huzii's method [9] determines the "winning" model from the set of all SE AR models

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to be the one with the smallest absolute third-order sample cumulant of $\widehat{u}(k)$ for lag $(k_1, k_2) = (0, -1)$. Tugnait [7] also proposed an optimization method by minimizing a cost function formed of the squared errors between theoretical output correlations and cumulants and the corresponding sample correlations and sample cumulants. Giannakis [10] proposed a method that converts the noncausal AR parameter estimation problem into a causal moving average (MA) parameter estimation problem.

In this correspodence, we propose a new parameter estimation method for a noncausal AR system $1/A(z)$ based on a new quadratic equation relating the unknown AR parameters to higher order cumulants of data. The proposed method finds the optimal estimate $1/\widehat{A}(z)$ through an iterative numerical optimization algorithm; it is applicable no matter whether or not the order of $1/A(z)$ is known in advance; it is also applicable for the case of causal AR model. Section II presents the new AR parameter estimation method. Then, some simulation results are provided to support the proposed AR parameter estimation method in Section III. Finally, we draw some conclusions.

II. A New Cumulant-Based Parameter Estimation Method for Noncausal AR Systems

Assume that $x(k)$ are the given noisy output measurements generated from a noncausal stable AR model as follows:

$$\sum_{i=-p_1}^{p_2} a(i)y(k-i) = u(k) \quad (1)$$

$$x(k) = y(k) + w(k) \quad (2)$$

where $u(k)$ is a real, zero-mean, independent identically distributed (i.i.d.) nonGaussian process with Mth-order (≥ 3) cumulant $\gamma_M \neq 0$, and $w(k)$ is Gaussian with unknown statistics. The pth-order ($p = p_1 + p_2$) noncausal AR system has a transfer function $H(z) = 1/A(z)$ where

$$A(z) = \sum_{k=-p_1}^{p_2} a(k)z^{-k} = A_1(z) \cdot A_2(z) \quad (3)$$

where

$$A_1(z) = a_1(-p_1)z^{p_1} + a_1(-p_1+1)z^{p_1-1} + \cdots + a_1(0) \quad (4)$$

is a p_1th-order polynomial of z with all the roots outside the unit circle (the anticausal part of $A(z)$) and

$$A_2(z) = a_2(0) + a_2(1)z^{-1} + \cdots + a_2(p_2)z^{-p_2} \quad (5)$$

is a p_2th-order polynomial of z^{-1} with all the roots inside the unit circle (the causal part of $A(z)$). Note that correlation-based AR spectral estimation methods can only provide an estimate of the SE minimum-phase $A_{\mathrm{MP}}(z)$ given by

$$A_{\mathrm{MP}}(z) = A_1(z^{-1}) \cdot A_2(z) \quad (6)$$

except for a scale factor.

Let $C_{M,x}(k_1, k_2, \ldots, k_{M-1})$ denote the Mth-order cumulant function of the nonGaussian stationary process $x(k)$. It has been shown in [1] that

$$\sum_{i=-p_1}^{p_2} a(i)C_{M,x}(i-k_1, \ldots, i-k_{M-1}) = \gamma_M h(-k_1)h(-k_2)\cdots h(-k_{M-1}) \quad (7)$$

where $h(k)$ is the impulse response of the noncausal AR system. Setting $k_1 = -k$ and $k_2 = k_3 = \cdots = k_{M-1} = 0$ in (7) yields

$$\sum_{i=-p_1}^{p_2} a(i)C_{M,x}(i+k, i, \ldots, i) = \sum_{i=-p_1}^{p_2} a(i)C_{M,x}(k, -i, 0, \ldots, 0) = \gamma_M h^{M-2}(0)h(k). \quad (8)$$

Convolving both sides of (8) with $a(k)$ gives rise to

$$\sum_{i=-p_1}^{p_2}\sum_{j=-p_1}^{p_2} a(i)a(j)C_{M,x}(k-j, -i, 0, \ldots, 0) = \mathbf{a}'\mathbf{C}(k)\mathbf{a} = \gamma_M h^{M-2}(0)\delta(k) \quad (9)$$

where $\delta(k)$ is the Kronecker delta function, $\mathbf{a}$ is a $p \times 1$ vector given by

$$\mathbf{a} = (a(-p_1), a(-p_1+1), \ldots, a(p_2))' \quad (10)$$

and $\mathbf{C}(k)$ is a $(p+1) \times (p+1)$ matrix whose (i,j)th component is given by

$$[\mathbf{C}(k)]_{i,j} = C_{M,x}(k + p_1 - j + 1, p_1 - i + 1, 0, \ldots, 0). \quad (11)$$

Assuming that p_1' and p_2' are chosen for p_1 and p_2, respectively, the proposed method based on the key quadratic equation (9) searches for the optimum $\mathbf{a}$ by minimizing a cost function of either $J = J_1$ or $J = J_2$ through an iterative numerical optimization algorithm where

$$J_1(\mathbf{a}) = \frac{\sum_{k=-K}^{K}(\mathbf{a}'\hat{\mathbf{C}}(k)\mathbf{a})^2}{(\mathbf{a}'\hat{\mathbf{C}}(0)\mathbf{a})^2} \geq 1 \quad (12)$$

and

$$J_2(\mathbf{a}) = \frac{\sum_{k=-K,k\neq 0}^{K}(\mathbf{a}'\hat{\mathbf{C}}(k)\mathbf{a})^2}{\|\mathbf{a}\|^4} \geq 0 \quad (13)$$

in which $\hat{\mathbf{C}}(k)$ is also a $(p+1) \times (p+1)$ matrix by replacing each component of $\mathbf{C}(k)$ with the associated Mth-order sample cumulant. It is almost impossible to find a closed-form solution for the optimum $\mathbf{a}$ because both J_1 and J_2 are highly nonlinear functions of $\mathbf{a}$. Instead, the proposed AR parameter-estimation method searches for the optimum $\mathbf{a}$ using a gradient type algorithm as follows:

($S1$) Set $t = 0$. Choose an initial guess for $\mathbf{a}_0 = [a(-p_1'), \ldots, a(p_2')]'$ and an objective function $J(\mathbf{a}) = J_1(\mathbf{a})$ or $J(\mathbf{a}) = J_2(\mathbf{a})$, and compute $J(\mathbf{a}_0)$.

($S2$) Set $t = t + 1$.

($S3$) Normalize $\mathbf{a}_t$ by $\mathbf{a}_t/\|\mathbf{a}_t\|$. Update $\mathbf{a}_t$ by

$$\mathbf{a}_t = \mathbf{a}_{t-1} - \rho\frac{\partial J(\mathbf{a}_{t-1})}{\partial \mathbf{a}_{t-1}} \quad (14)$$

such that $J(\mathbf{a}_t) < J(\mathbf{a}_{t-1})$, where ρ is a positive constant.

($S4$) If

$$\frac{J(\mathbf{a}_{t-1}) - J(\mathbf{a}_t)}{J(\mathbf{a}_{t-1})} \geq \varepsilon \quad (15)$$

where ε is a preassigned small positive constant, then go to ($S2$); otherwise, stop.

Some worthy remarks regarding the proposed AR parameter estimation method are summarized in the following:

(R1) The typical value for ρ in updating $\mathbf{a}_t$ with (14) is $\rho = 1$. However, when updating $\mathbf{a}_t$ with (14) results in $J(\mathbf{a}_t) \geq J(\mathbf{a}_{t-1})$, one may continually decrease the value for ρ by $\rho/2$ until the associated $J(\mathbf{a}_t) < J(\mathbf{a}_{t-1})$.

(R2) The proposed AR parameter estimation method is a single-step nonlinear optimization algorithm that fits the key quadratic equation given by (9) with Mth-order sample cumulants of nonGaussian measurements such that either J_1 or J_2 is minimum. It relies on neither the SE $A_{\mathrm{MP}}(z)$

as an exhaustive search method [7], [9], nor the MP-AP decomposition based method [8], nor any conversion procedure as in Giannakis' method [10]. However, the optimum solution for $\mathbf{a}$ is not resolvable to a scale factor since $J_1(\mathbf{a}) = J_1(b\mathbf{a})$ and $J_2(\mathbf{a}) = J_2(b\mathbf{a})$ for any $b \neq 0$.

(R3) When p_1 and p_2 are known in advance, the proposed method works well, whereas the objective function J_2 is preferred to J_1 because it is less sensitive to initial conditions for $\mathbf{a}$ by our experience.

(R4) When $(p_1' \neq p_1)$ and $(p_2' \neq p_2)$ but $(p_1'+p_2' = p_1 - p_2 = p)$ is known, the optimum $A(z)$ turns out to be an estimate $\widehat{A}(z) = \alpha A(z) \cdot z^{-\tau}$, where $\tau = p_1 - p_1' = p_2' - p_2$ because cumulants are blind to time delay factors. However, $\mathbf{a}'\hat{\mathbf{C}}(0)\mathbf{a} = \gamma_M[\widehat{h}(0)]^{M-2}$ (the denominator of J_1) could equal zero since $\widehat{h}(k) \approx (1/\alpha)h(k+\tau)$, and therefore, J_2 is preferred to J_1 for this case.

(R5) When none of p_1, p_2, and p are known *a priori*, the optimum estimate $\widehat{A}(z)$ with $p_1' \geq p_1$ and $p_2' \geq p_2$ turns out to be an estimate of $\alpha A(z) \cdot z^{-\tau}$, where $p_1 - p_1' \leq \tau \leq p_2' - p_2$. However, we empirically found that the proposed method associated with J_1 always provides an optimum estimate $\widehat{A}(z) \cong \alpha A(z)$ for noncausal AR systems with $\max|h(k)| = |h(0)|$. The reason for this is that the minimum value of J_1 for $\tau = 0$ is always smaller than that for $\tau \neq 0$ because the value of $\mathbf{a}'\mathbf{C}(0)\mathbf{a} = \gamma_M \widehat{h}(0)^{M-2}\gamma_M(h(\tau)/\alpha)^{M-2}$ for $\tau = 0$ is larger than that for $\tau \neq 0$ in absolute value (see (12)).

(R6) The proposed AR parameter estimation method is applicable for both causal and noncausal AR systems as long as $\gamma_M \neq 0$ for any $M \geq 3$ because the causal AR(p) model is nothing but a special case of a noncausal AR model for $p_1 = 0$ and $p_2 = p$.

Next, let us show some simulation results to support the proposed parameter estimation method for noncausal AR systems.

III. Simulation Examples

Two simulation examples are to be presented. Example 1 assumes that (p_1, p_2) are known *a priori*, and Example 2 deals with the case that $p_1' \geq p_1$ and $p_2' \geq p_2$, i.e., the order of the AR system is overdetermined. In the simulation, the driving input $u(k)$ used was a zero-mean exponentially distributed i.i.d. random sequence with variance $\sigma_u^2 = 1$ and skewness $\gamma_3 = 2$, and for the selected noncausal AR model $1/A(z) = 1/[A_1(z)A_2(z)]$ (see (3)), data $x(k)$ of length $N = 1024$ were generated by letting $u(k)$ pass through the causal stable AR system (forward processing) $1/A_2(z)$ followed by the anticausal stable system (backward processing) $1/A_1(z)$ for three different signal-to-noise ratios (SNR) (10, 50, 100), where noise $w(k)$ was white Gaussian. Mean and standard deviation were calculated from 30 independent estimates of $\widehat{A}(z)$ with $\sum_k \widehat{a}^2(k) = \sum_k a^2(k) = 1$ obtained by the proposed method with cumulant order $M = 3$ and $K = 15$ in J_1 and J_2. With the same simulation data, the corresponding results were also obtained using Huzii's method in Example 1 for performance comparison. Next, let us turn to Example 1.

Example 1: Two cases for both p_1 and p_2 known *a priori* are considered as follows:

Case 1: $p_1 = 1$ and $p_2 = 2$.

$$A(z) = -0.3413z + 0.6996 - 0.5631z^{-1} + 0.2773z^{-2}$$
$$\text{Poles}: 0.4 \pm 0.7j, 1.25$$
$$\text{Initial guess: } \mathbf{a}_0 = [0\ 1\ 0\ 0]'. \tag{16}$$

TABLE I
Simulation Results for Case 1 of Example 1

True (normalized) AR parameters: $a(-1) = -0.3413$, $a(0) = 0.6996$, $a(1) = -0.5631$, $a(2) = 0.2773$				
Method	Estimated AR parameters (mean±standard deviation)			
		$SNR = 100$	$SNR = 50$	$SNR = 10$
The proposed method ($J = J_2$)	$\widehat{a}(-1)$	−0.4303±0.0501	−0.4428±0.0418	−0.3949±0.1248
	$\widehat{a}(0)$	0.6695±0.0170	0.6656±0.0150	0.6385±0.0565
	$\widehat{a}(1)$	−0.5600±0.0170	−0.5592±0.0202	−0.5944±0.0844
	$\widehat{a}(2)$	0.2207±0.0349	0.2119±0.0293	0.2334±0.0593
Huzii's method	$\widehat{a}(-1)$	−0.3496±0.0126	−0.3515±0.0170	−0.1712±0.1127
	$\widehat{a}(0)$	0.7235±0.0083	0.7460±0.0096	0.5289±0.2775
	$\widehat{a}(1)$	−0.5348±0.0107	−0.5110±0.0134	−0.5764±0.3669
	$\widehat{a}(2)$	0.2606±0.0119	0.2409±0.0148	0.1759±0.3387

TABLE II
Simulation Results for Case 2 of Example 1

True (normalized) AR parameters: $a(-3) = -0.2071$, $a(-2) = 0.3659$, $a(-1) = -0.5247$, $a(0) = 0.6317$, $a(1) = -0.3452$, $a(2) = 0.1726$				
Method	Estimated AR parameters (mean±standard deviation)			
		$SNR = 100$	$SNR = 50$	$SNR = 10$
The proposed method ($J = J_2$)	$\widehat{a}(-3)$	−0.2119±0.0170	−0.2143±0.0183	−0.2611±0.0587
	$\widehat{a}(-2)$	0.3632±0.0132	0.3608±0.0155	0.3131±0.0768
	$\widehat{a}(-1)$	−0.5239±0.0121	−0.5212±0.0142	−0.4710±0.0670
	$\widehat{a}(0)$	0.6330±0.0112	0.6370±0.0127	0.6910±0.0575
	$\widehat{a}(1)$	−0.3423±0.0174	−0.3400±0.0215	−0.2946±0.0734
	$\widehat{a}(2)$	0.1720±0.0149	0.1711±0.0167	0.1548±0.0326
Huzii's method	$\widehat{a}(-3)$	−0.2236±0.0242	−0.2017±0.0950	−0.1276±0.1931
	$\widehat{a}(-2)$	0.3926±0.0445	0.3648±0.1547	0.2317±0.2483
	$\widehat{a}(-1)$	−0.5475±0.0177	−0.5397±0.0332	0.0740±0.2201
	$\widehat{a}(0)$	0.6379±0.0263	0.6289±0.0454	−0.2554±0.3461
	$\widehat{a}(1)$	−0.2556±0.0419	−0.2317±0.1146	0.4094±0.2386
	$\widehat{a}(2)$	0.1335±0.0299	0.1541±0.1223	−0.5090±0.3502

Case 2: $p_1 = 3$ and $p_2 = 2$.

$$A(z) = -0.2071z^3 + 0.3659z^2 - 0.5247z + 0.6317 - 0.3452z^{-1} + 0.1726z^{-2}$$
$$\text{Poles}: -0.02 \pm 1.1752j, 0.3 \pm 0.6403j, 1.206$$
$$\text{Initial guess: } \mathbf{a}_0 = [0\ 0\ 0\ 1\ 0\ 0]'. \tag{17}$$

The objective function J_2 with $p_1' = p_1$ and $p_2' = p_2$ was used, and the parameter ρ and ε were set to $\rho = 1$ and $\varepsilon = 1 \times 10^{-7}$ in the associated gradient-type optimization algorithm. For Huzii's method, the SE minimum-phase $\widehat{A}_{\mathrm{MP}}(z)$ was obtained by the well-known Burg's method [17], and the desired $\widehat{A}(z)$ was determined to be the one with minimum $\{|C_{3,\widehat{u}}(0,-1)| + |C_{3,\widehat{u}}(0,-2)| + \cdots + |C_{3,\widehat{u}}(0,-5)|\}\ /|C_{3,\widehat{u}}(0,0)|$, where $\widehat{u}(k)$ is the associated output of $\widehat{A}(z)$ with the input $x(k)$. The simulation results obtained using both methods are shown in Tables I and II for the above two cases, respectively, together with the true third-order noncausal AR parameters. The typical number of iterations spent by the optimization algorithm ranges from 30 to 100 for Case 1 and from 40 to 120 for Case 2. From these tables, one can see that both methods have smaller variance and bias for higher SNR. For SNR = 100, both the proposed method and Huzii's method have similar performance, whereas for SNR = 10, the performance of the former is much better than that of the latter. The reason for this is that the proposed cumulant-based AR parameter estimation method is Gaussian noise insensitive, but Huzii's method relies on an accurate estimate for $A_{\mathrm{MP}}(z)$, which was obtained by a noise-sensitive correlation-based AR spectrum estimation algorithm. Inaccurate estimates $\widehat{A}_{\mathrm{MP}}(z)$ provided incorrect possible pole locations of the AR system and further led to incorrect decisions in determining the noncausal SE

TABLE III
SIMULATION RESULTS OF EXAMPLE 2

Objective function: $J = J_1$; $p_1 = 2$, $p_2 = 0$, $p'_1 = 2$, $p'_2 = 2$; True (normalized) AR parameters: $a(-2) = 0.6271$, $a(-1) = 0.3484$, $a(0) = 0.6967$, $a(1) = 0$, $a(2) = 0$			
Estimated AR parameters (mean±standard deviation)			
	SNR=100	SNR=50	SNR=10
$\hat{a}(-2)$	0.6311±0.0287	0.6346±0.0325	0.6546±0.0504
$\hat{a}(-1)$	0.3339±0.0391	0.3282±0.0440	0.2500±0.1092
$\hat{a}(0)$	0.6930±0.0313	0.6898±0.0386	0.6304±0.1537
$\hat{a}(1)$	−0.0134±0.0565	−0.0233±0.0660	−0.1422±0.1652
$\hat{a}(2)$	−0.0022±0.0567	−0.0050±0.0656	−0.0546±0.1297

candidate $A(z)$, which accounts for the performance degradation. This also indicates that the proposed method is much more insensitive to Gaussian noise than Huzii's method. Globally speaking, these simulation results support the good performance of the proposed method.

Example 2: None of $p_1 = 2, p_2 = 0$, and $p = p_1 + p_2 = 2$ are known.

$$A(z) = 0.6271z^2 + 0.3484z + 0.6967$$
$$\text{Poles: } -0.2778 \pm 1.0168j$$
$$\text{Initial guess: } \mathbf{a}_0 = [0\ 0\ 1\ 0\ 0]'. \tag{18}$$

The objective function J_1 with $p'_1 = 2$ $(\geq p_1 = 2)$ and $p'_2 = 2$ $(\geq p_2 = 0)$ was used, and the parameters ρ and ε were also set to $\rho = 1$ and $\varepsilon = 1 \times 10^{-7}$ in the associated gradient type optimization algorithm. The simulation results, together with the true second-order anticausal AR parameters, are shown in Table III. The typical number of iterations spent by the optimization algorithm ranges from 10 to 50 for this example. Note that $h(k) = 0$ for $k > 0$ and $\max|h(k)| = |h(0)| = 1.4353$ for this case. From Table III, one can see that estimates $\widehat{a}(-2)$, $\widehat{a}(-1)$, and $\widehat{a}(0)$ are quite close to $a(-2)$, $a(-1)$, and $a(0)$, respectively, and $\widehat{a}(1)$ and $\widehat{a}(2)$ are around zero. These simulation results also justify the statements presented in (R5).

IV. CONCLUSION

We have presented a new cumulant-based parameter estimation method for noncausal AR systems based on a new quadratic equation given by (9). The proposed method is a nonlinear estimation algorithm minimizing either J_1 given by (12) or J_2 given by (13), and the unknown pth-order AR noncausal system $1/A(z)$ where $A(z)$ is given by (3) can be estimated except for a scale factor (see (R2)). When both p_1 and p_2 are known, J_2 is preferred to J_1 (see (R3)); otherwise, an unknown time delay may exist in the estimated $A(z)$; when $p = p_1 + p_2$ is known but p_1 and p_2 are not known, J_2 is also preferred to J_1 (see (R4)); when none of p_1, p_2, and p are known, J_1 is preferred to J_2 with $p'_1 \geq p_1$ and $p'_2 \geq p_2$ (see (R5)). The proposed method is also applicable for the case of causal AR model (see (R6)). Finally, two simulation examples were provided to justify that the proposed AR parameter estimation method is effective. The presented simulation examples also indicate that the proposed method is much more insensitive to Gaussian noise than those methods reported in [7]–[9] that rely on the estimation of SE $A_{\mathrm{MP}}(z)$ by noise-sensitive correlation-based methods.

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