Correspondence

A Further Analysis for the Minimum-Variance Deconvolution Filter Performance

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***Abstract*—Chi and Mendel [1], [9] analyzed the performance of the minimum-variance deconvolution (MVD). In this correspondence, a further analysis of the performance of the MVD filter is presented. We show that the MVD filter performs like an inverse filter and a whitening filter as SNR → ∞, and like a matched filter as SNR → 0. The estimation error $e(k)$ of the MVD filter is colored noise, but it becomes white when SNR → 0. This analysis also connects the error power spectral density of the MVD filter with the spectrum of the causal prediction error filter.**

I. Introduction

Mendel [2], [3], [6] developed a minimum-variance deconvolution (MVD) filter for the following linear time-invariant convolution model:

$$z(k) = \mu(k) * v(k) + n(k)$$
$$= \sum_{i=0}^{k} v(i)\,\mu(k-i) + n(k). \tag{1}$$

In this model, $v(k)$, $k = 0, 1, 2, \cdots$, is the impulse response sequence of the signal distorting system (e.g., impulse response of communication channel, seismic source wavelet), $\mu(k)$ is the desired signal sequence (e.g., message, reflectivity sequence), and $n(k)$ is the measurement noise, which accounts for physical effects not explained by the noise-free signal $\mu(k) * v(k)$ as well as sensor noise. They assume that $\mu(k)$ and $n(k)$ are white and zero mean, with variances

$$E[\mu^2(k)] = \sigma_\mu^2 \tag{2}$$

and

$$E[n^2(k)] = R, \tag{3}$$

respectively.

The fixed-interval MVD filter estimates the input signal $\mu(k)$ from a set of measurements $z(k)$, $k = 1, 2, \cdots, N$ where N is the total number of measurements. A well-known fact is that the linear minimum-variance estimate $\hat{\boldsymbol{\mu}}$ is given by [4]–[6]

$$\hat{\boldsymbol{\mu}} = \sigma_\mu^2 V' \Omega^{-1} z \tag{4}$$

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where

$$z = \text{col}\,(z(1), z(2), \cdots, z(N)), \tag{5}$$

$$V = \begin{pmatrix} v(0) & 0 & \cdots & 0 \\ v(1) & v(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v(N-1) & v(N-2) & \cdots & v(0) \end{pmatrix}, \tag{6}$$

and Ω is the covariance matrix

$$\Omega = E[zz'] = \sigma_\mu^2 VV' + RI \tag{7}$$

with I being an identity matrix. The covariance matrix of estimation error $e(k)$, defined by

$$e(k) = \mu(k) - \hat{\mu}(k) \tag{8}$$

is

$$E[ee'] = \sigma_\mu^2 I - \sigma_\mu^4 V'\Omega^{-1} V. \tag{9}$$

Chi and Mendel derived the frequency response $V_{MV}(\omega)$ of the steady-state MVD filter to analyze its performance. From the derived $V_{MV}(\omega)$, they found that the steady-state MVD filter is equivalent to Berkhout's two-sided least-squares filter [8]. They quantitatively showed the undershoot and zero-phase properties of the MVD filter output based on $V_{MV}(\omega)$. They also showed that the error variance $E[e(k)^2]$ depends on the signal-to-noise ratio (SNR) defined by

$$\text{SNR} = \frac{\sigma_\mu^2}{R} \sum_{k=0}^{\infty} v^2(k) \tag{10}$$

and only slightly on data length. In this correspondence, we present a further analysis of the behavior of the MVD filter. In Section II, we derive the power spectral density $\Phi_e(\omega)$ of $e(k)$. In Section III, we present some properties about the behavior of the MVD filter based on the theoretical results obtained in Section II. We draw conclusions in Section IV.

II. Power Spectral Density of the Estimation Error

We begin with the derivation of the autocorrelation function of the output of the MVD filter. Let

$$\phi_{\hat{\mu}}(k, k+l) = E[\hat{\mu}(k)\,\hat{\mu}(k+l)] = \sigma_\mu^4 v_k' \Omega^{-1} v_{k+l} \tag{11}$$

where v_k is the kth column of the matrix V. Note, from (11), that $\phi_{\hat{\mu}}(k, k+l)$ depends not only on l but also on k. When the MVD filter reaches the steady state, it becomes a linear time-invariant filter with the frequency response $V_{MV}(\omega)$ being [1], [9]

$$V_{MV}(\omega) = \frac{\sigma_\mu^2 V^*(\omega)}{\sigma_\mu^2 |V(\omega)|^2 + R} \tag{12}$$

where $V(\omega)$ is the Fourier transform of $v(k)$ and $V^*(\omega)$ is the complex conjugate of $V(\omega)$. Therefore, the output $\mu(k)$ of the MVD filter becomes a wide sense stationary random process when the MVD filter reaches the steady state. In the steady state,

$$\hat{\mu}(k) = z(k) * v_{MV}(k) = \mu(k) * v_{out}(k) + n(k) * v_{MV}(k) \tag{13}$$

where

$$v_{out}(k) = v_{MV}(k) * v(k), \tag{14}$$

whose frequency response is

$$V_{out}(\omega) = \frac{\sigma_\mu^2 |V(\omega)|^2}{\sigma_\mu^2 |V(\omega)|^2 + R}. \tag{15}$$

The autocorrelation function $\phi_{\hat{\mu}}(k, k + l)$ depends only on l when the MVD filter reaches the steady state. In other words,

$$\phi_{\hat{\mu}}(k, k + l) = \phi_{\hat{\mu}}(l). \tag{16}$$

From (12)–(15), we can see that the power spectral density $\Phi_{\hat{\mu}}(\omega)$ of $\hat{\mu}(k)$ is

$$\Phi_{\hat{\mu}}(\omega) = \sigma_\mu^2 |V_{out}(\omega)|^2 + R|V_{MV}(\omega)|^2 = \frac{\sigma_\mu^4 |V(\omega)|^2}{\sigma_\mu^2 |V(\omega)|^2 + R} = \sigma_\mu^2 V_{out}(\omega). \tag{17}$$

From (9) and (11), we see that the correlation function of the estimation error $e(k)$ is

$$\phi_e(l) = \sigma_\mu^2 \delta(l) - \phi_{\hat{\mu}}(l) \tag{18}$$

when the MVD filter is in the steady state. Therefore, from (17) and (18), we have

$$\Phi_e(\omega) = \frac{\sigma_\mu^2 R}{\sigma_\mu^2 |V(\omega)|^2 + R}, \tag{19}$$

which implies that $e(k)$ is colored noise. In the next section, we present the behavior of the MVD filter based on the above results.

III. Behavior of the MVD Filter

We now present the properties from $\Phi_e(\omega)$ and $V_{out}(\omega)$ obtained in Section II in the following:

1)

$$\Phi_e(\omega) \to \frac{R}{|V(\omega)|^2}$$

and

$$V_{out}(\omega) \to 1$$

as $R \to 0$ or $\sigma_\mu^2 \to \infty$ (i.e., SNR $\to \infty$).

2)

$$\Phi_e(\omega) \to \sigma_\mu^2$$

and

$$V_{out}(\omega) \to \sigma_\mu^2 \frac{|V(\omega)|^2}{R}$$

as $R \to \infty$ or $\sigma_\mu^2 \to 0$ (i.e., SNR $\to 0$).

3) The error variance is $\sigma^2 = \phi_e(0) = \sigma_\mu^2(1 - v_{out}(0))$, which was mentioned in [1] without proof.

4) Because

$$\phi_{\hat{\mu}}(k) = \sigma_\mu^2 v_{out}(k)$$

[see (17)], the autocorrelation function of the output of the MVD filter has the same waveshape as its output wavelet $v_{out}(k)$.

5) $\phi_e(k)$ is proportional to the autocorrelation function $\phi_{pef}(k)$ of the impulse response of the prediction error filter [10] because

$$\Phi_{pef}(\omega) = \frac{\eta}{\sigma_\mu^2 |V(\omega)|^2 + R}$$

where η is the error variance of the prediction error filter.

It is well known that the prediction error filter is a causal whitening filter no matter what the SNR is. Unlike the prediction error filter, the MVD filter performs differently. We make the following conclusions about the behavior of the MVD filter.

The property 1) implies that a) the MVD filter performs like an inverse filter and b) the estimation error $e(k)$ is colored noise when SNR $\to \infty$.

The properties 1) and 4) imply that the MVD filter also performs like a whitening filter when SNR $\to \infty$ because the output of the MVD filter approaches white noise.

The property 2) implies that the MVD filter performs like a matched filter and $e(k)$ is white noise when SNR $\to 0$.

The property 5) implies that the impulse response of the prediction error filter can be obtained by solving a set of normal equations using only the measurement $z(k)$. On the other hand, R, σ_μ^2, and $V(k)$ must be given ahead of time when the MVD filter is used. From the impulse response of the prediction error filter, one can predict the shape of the error power spectral density of the MVD filter, from which one may further infer some information about SNR. For example, if the obtained $\Phi_{pef}(\omega)$ for given $z(k)$ is very flat (a constant), then $\Phi_e(\omega)$ is also flat. From the property 2), we predict that the SNR is small.

IV. Conclusions

In this correspondence, we began with the deviation of the power spectral density of the output of the steady-state MVD filter. Based on the derived results, we presented some properties about the behavior of the MVD filter. The MVD filter performs like an inverse filter and a whitening filter as SNR $\to \infty$. It performs like a matched filter as SNR $\to 0$. The estimation error of the MVD filter is colored noise, but it becomes white when SNR $\to 0$. The error power spectral density of the MVD filter also provides a connection with the spectrum of the causal prediction error filter. From this connection, some information about SNR may be obtained from the measurement $z(k)$.

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