SOME RESULTS ON 16-QAM MIMO DETECTION USING SEMIDEFINITE RELAXATION

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ABSTRACT

Semidefinite relaxation (SDR) is a high-performance efficient approach to MIMO detection especially for the BPSK or QPSK constellations. Recently, a number of research endeavors have focused on extending SDR to the case of 16-QAM constellations. This paper reports two interesting and useful results on this problem. First, we show that two of the existing 16-QAM SDR receivers, namely the polynomial-inspired SDR (PI-SDR) and bound-constrained SDR (BC-SDR) methods, are equivalent. Second, we develop a specialized interior-point algorithm for the implementation of BC-SDR. The proposed algorithm is computationally efficient exploiting the BC-SDR structures, and enables us to handle larger problem sizes in practice.

Index Terms— MIMO detection, semidefinite relaxation, convex optimization

1. INTRODUCTION

Multiple-input-multiple-output (MIMO) detection using semidefinite relaxation (SDR) [1–6] has recently received increasing attention. Being a constellation dependent technique, SDR has been shown to provide considerably better symbol error performance than the linear and decision-feedback MIMO receivers. Though not an optimal maximum-likelihood (ML) receiver, SDR guarantees a worst-case polynomial-time complexity in the number of inputs. In comparison, the currently best known optimal ML implementations, namely sphere decoding [7, 8], do not have such a guarantee [9] and would be too expensive to employ for large number of inputs. (Note that the above argument may be inapplicable to suboptimal ML variants of sphere decoding.)

SDR was first proposed for the BPSK and QPSK constellations [1, 2], in which cases near-optimal performance was empirically observed. Very recently, a rigorous theoretical study has confirmed that BPSK SDR can actually achieve the full diversity [10]. An extension to MPSK has been reported in [3]. Presently there are several concurrent works competing in the case of 16-QAM constellations. The first endeavor of 16-QAM SDR is the polynomial-inspired SDR (PI-SDR) method by Wiesel *et al.* [4]. Interestingly, PI-SDR is shown to be a bidual of the ML (or achieves an optimal Lagrangian dual lower bound of the ML). Later, Sidropoulos *et al.* proposed a bound-constrained SDR (BC-SDR) method [5] that has the simplest structures among the various 16-QAM SDR methods. Mobasher *et al.* formulated a class of SDR problems that is applicable to any kind of symbol constellations [6]. Mobasher's approach is sophisticated, but its complexity requirement is also the highest among the various SDR methods.

The contributions of this paper are twofold. First, we prove that PI-SDR and BC-SDR are indeed equivalent. Specifically, the optimal values of the two SDR problems are shown to be identical. Second, the implementations of the existing 16-QAM SDR works rely on general-purpose solvers, such as SeDuMi [11]. We develop a specialized solver for BC-SDR that runs many times faster than general-purpose solvers. This newly developed solver enables us to handle larger problem sizes, such as the 40×40 16-QAM system in which no test has been conducted in the previous SDR works.

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2. PROBLEM FORMULATION

We consider a standard MIMO detection problem in which the received signal is modeled as

$$\tilde{\mathbf{y}} = \tilde{\mathbf{H}}\tilde{\mathbf{s}} + \tilde{\mathbf{v}} \quad (1)$$

where $\tilde{\mathbf{y}} \in \mathbb{C}^{\tilde{M}}$ is the received vector, $\tilde{\mathbf{s}} \in \mathbb{C}^{\tilde{N}}$ is the transmitted symbol vector, $\tilde{\mathbf{H}} \in \mathbb{C}^{\tilde{M}\times\tilde{N}}$ is the MIMO channel, $\tilde{\mathbf{v}}$ is complex circular additive white Gaussian noise, $\tilde{M}$ is the number of receiver antennas, and $\tilde{N}$ is the number of transmitter antennas. Each element of $\tilde{\mathbf{s}}$ is drawn from a constellation set, denoted by $\mathcal{S}$.

Here, our emphasis is placed on the 16-QAM constellations where $\mathcal{S} = \{\, s = s_R + js_I \mid s_R, s_I \in \{\pm 1, \pm 3\}\,\}$. Let us define

$$\mathbf{y} = \begin{bmatrix} \Re\{\tilde{\mathbf{y}}\} \\ \Im\{\tilde{\mathbf{y}}\} \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} \Re\{\tilde{\mathbf{s}}\} \\ \Im\{\tilde{\mathbf{s}}\} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \Re\{\tilde{\mathbf{H}}\} & -\Im\{\tilde{\mathbf{H}}\} \\ \Im\{\tilde{\mathbf{H}}\} & \Re\{\tilde{\mathbf{H}}\} \end{bmatrix},$$

$M = 2\tilde{M}$, and $N = 2\tilde{N}$. The complex-valued $\tilde{M} \times \tilde{N}$ 16-QAM signal model in (1) can be equivalently represented by a real-valued, virtually $M \times N$ 4-PAM model, given as follows

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{v} \quad (2)$$

where $\mathbf{s} \in \{\pm 1, \pm 3\}^N$ and $\mathbf{v}$ is defined in the same way as $\mathbf{y}$.

The following is the ML detection problem of (2):

$$\min_{\mathbf{s}\in\{\pm 1,\pm 3\}^N} \|\mathbf{y} - \mathbf{H}\mathbf{s}\|_2^2 \quad (3)$$

The (globally) optimal solution of (3), or the ML decision, provides superior detection performance. However, it is not easy at all to solve (3) optimally. The currently best known optimal ML approach is sphere decoding [7], which has been practically found to be computationally very attractive for small to moderate N (say, for $N \leq 20$). However, it is now understood that the sphere decoding complexity would become prohibitive for large N [9].

3. REVIEW OF TWO 16-QAM SDR DETECTORS

SDR is a suboptimal ML approach based on semidefinite programming. Unlike sphere decoding, the SDR complexity is polynomial in N. Here we review two 16-QAM SDR MIMO detection methods, namely PI-SDR [4] and BC-SDR [5].

Let us consider the following problem which can be verified to be equivalent to the ML problem in (3):

$$\begin{aligned} \min_{\mathbf{S}\in\mathbb{S}^N,\mathbf{s}\in\mathbb{R}^N} \quad & \mathrm{tr}(\mathbf{H}^T\mathbf{H}\mathbf{S}) - 2\mathbf{s}^T\mathbf{H}^T\mathbf{y} + \|\mathbf{y}\|_2^2 \\ \text{s.t.} \quad & \mathbf{S} = \mathbf{s}\mathbf{s}^T, \quad S_{ii} \in \{1,9\}, \quad i = 1,\ldots,N \end{aligned} \tag{4}$$

where $\mathbf{S}$ is a slack variable, $\mathbb{S}^N$ is the set of $N \times N$ real symmetric matrices, S_{ij} is the (i,j)th element of $\mathbf{S}$, and $\mathrm{tr}(\cdot)$ is the trace operator. In BC-SDR, we consider a relaxation of (4) given by

$$\begin{aligned} \min \quad & \mathrm{tr}(\mathbf{H}^T\mathbf{H}\mathbf{S}) - 2\mathbf{s}^T\mathbf{H}^T\mathbf{y} + \|\mathbf{y}\|_2^2 \\ \text{s.t.} \quad & \mathbf{S} \succeq \mathbf{s}\mathbf{s}^T, \quad 1 \le S_{ii} \le 9, \quad i = 1,\ldots,N \end{aligned} \tag{5}$$

(In the above equation, we assume the tacit understanding that $\mathbf{S}$ and $\mathbf{s}$ are the optimization variables.) Here, the discrete constraints $S_{ii} \in \{1,9\}$ are relaxed to $1 \le S_{ii} \le 9$, and a semidefinite relaxation $\mathbf{S} \succeq \mathbf{s}\mathbf{s}^T$ is used in place of the nonconvex constraint $\mathbf{S} = \mathbf{s}\mathbf{s}^T$ [where $\mathbf{A} \succeq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is positive semidefinite (PSD)]. The resultant problem in (5) is a semidefinite program (SDP), whose globally optimal solution can be efficiently obtained by available interior-point methods [11, 12].

Once an optimal solution associated with $(\mathbf{S}, \mathbf{s})$ is obtained, our next step is to use the solution to approximate the ML solution. For example, we can simply round $\mathbf{s}$ to the nearest point in $\{\pm 1, \pm 3\}^N$ which requires only a straightforward quantization. Another method that is more popular in practice is the Gaussian randomization; see [2, 5] for the details.

PI-SDR takes on a rather different direction. It uses the following polynomial relation

$$u \in \{1,9\} \Longleftrightarrow (u-1)(u-9) = 0 \Longleftrightarrow u^2 - 10u + 9 = 0$$

to reformulate the ML problem in (4) as

$$\begin{aligned} \min_{\mathbf{S},\mathbf{s},\mathbf{U},\mathbf{u}} \quad & \mathrm{tr}(\mathbf{H}^T\mathbf{H}\mathbf{S}) - 2\mathbf{s}^T\mathbf{H}^T\mathbf{y} + \|\mathbf{y}\|_2^2 \\ \text{s.t.} \quad & \mathbf{S} = \mathbf{s}\mathbf{s}^T, \quad \mathbf{U} = \mathbf{u}\mathbf{u}^T \\ & d(\mathbf{S}) = \mathbf{u}, \quad d(\mathbf{U}) - 10\mathbf{u} + 9\mathbf{1}_N = \mathbf{0} \end{aligned} \tag{6}$$

where $d : \mathbb{R}^{N\times N} \to \mathbb{R}^N$ is the diagonal operator that extracts the main diagonals of its input to form a vector, and $\mathbf{1}_N$ stands for the N-dimensional all-one vector. PI-SDR is the semidefinite relaxation of (6), given by

$$\begin{aligned} \min \quad & \mathrm{tr}(\mathbf{H}^T\mathbf{H}\mathbf{S}) - 2\mathbf{s}^T\mathbf{H}^T\mathbf{y} + \|\mathbf{y}\|_2^2 \\ \text{s.t.} \quad & \mathbf{S} \succeq \mathbf{s}\mathbf{s}^T, \quad \mathbf{U} \succeq \mathbf{u}\mathbf{u}^T \\ & d(\mathbf{S}) = \mathbf{u}, \quad d(\mathbf{U}) - 10\mathbf{u} + 9\mathbf{1}_N = \mathbf{0} \end{aligned} \tag{7}$$

Again, (7) is an SDP and the treatment after the relaxation step is somewhat similar to that in BC-SDR.

It is interesting to compare the two SDR methods. Currently it is known that

1. PI-SDR is more expensive to implement than BC-SDR, because the former contains more optimization variables [5].
2. PI-SDR is a bidual of the ML problem [4], a desirable property from a standpoint of Lagrangian dual theory. It is not known whether BC-SDR has such a property.

Though the two SDR methods appear to be different, we will show in the next section that they are indeed equivalent.

4. RELATIONSHIP OF THE TWO SDRS

To prove the relationship of PI-SDR and BC-SDR, let

$$\begin{aligned} f_{\mathsf{BC-SDR}} = \min \ & \mathrm{tr}(\mathbf{H}^T\mathbf{H}\mathbf{S}) - 2\mathbf{s}^T\mathbf{H}^T\mathbf{y} + \|\mathbf{y}\|_2^2 \\ \text{s.t.} \ & \mathbf{S} \succeq \mathbf{s}\mathbf{s}^T, \ 1 \le S_{ii} \le 9, \ i = 1,\ldots,N \end{aligned} \tag{8}$$

$$\begin{aligned} f_{\mathsf{PI-SDR}} = \min \ & \mathrm{tr}(\mathbf{H}^T\mathbf{H}\mathbf{S}) - 2\mathbf{s}^T\mathbf{H}^T\mathbf{y} + \|\mathbf{y}\|_2^2 \\ \text{s.t.} \ & \mathbf{S} \succeq \mathbf{s}\mathbf{s}^T, \quad \mathbf{U} \succeq \mathbf{u}\mathbf{u}^T \\ & d(\mathbf{S}) = \mathbf{u}, \quad d(\mathbf{U}) - 10\mathbf{u} + 9\mathbf{1}_N = \mathbf{0} \end{aligned} \tag{9}$$

denote the optimal values of BC-SDR in (5) and PI-SDR in (7), respectively. Consider the following proposition:

Proposition 1 *The bound-constrained SDR and polynomial-inspired SDR problems achieve the same optimal value; i.e.,*

$$f_{\mathsf{BC-SDR}} = f_{\mathsf{PI-SDR}}. \tag{10}$$

Proof: First, we prove that $f_{\mathsf{BC-SDR}} \le f_{\mathsf{PI-SDR}}$. Suppose that $(\tilde{\mathbf{S}}, \tilde{\mathbf{s}}, \tilde{\mathbf{U}}, \tilde{\mathbf{u}})$ is an optimal point of (9). The feasibility condition $\tilde{\mathbf{U}} - \tilde{\mathbf{u}}\tilde{\mathbf{u}}^T \succeq \mathbf{0}$ implies that $\tilde{U}_{ii} - \tilde{u}_i^2 \ge 0$ for all i. Subsequently,

$$0 = \tilde{U}_{ii} - 10\tilde{u}_i + 9 \ge \tilde{u}_i^2 - 10\tilde{u}_i + 9 = (\tilde{u}_i - 1)(\tilde{u}_i - 9)$$

for all i. The inequality above is equivalent to $(\tilde{S}_{ii}-1)(\tilde{S}_{ii}-9) \le 0$, or $1 \le \tilde{S}_{ii} \le 9$. Thus, $(\tilde{\mathbf{S}}, \tilde{\mathbf{s}})$ is feasible to (8) yielding an objective value equal to $f_{\mathsf{PI-SDR}}$.

Second, we prove that $f_{\mathsf{BC-SDR}} \ge f_{\mathsf{PI-SDR}}$. Suppose that $(\hat{\mathbf{S}}, \hat{\mathbf{s}})$ is an optimal point of (8). Let $\hat{\mathbf{u}} = d(\hat{\mathbf{S}})$, and

$$\hat{\mathbf{U}} = \hat{\mathbf{u}}\hat{\mathbf{u}}^T + D(\mathbf{w}) \tag{11}$$

where $D : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is the operator that outputs a diagonal matrix with its main diagonals being the input, and $\mathbf{w}$ is given by

$$w_i = -(\hat{S}_{ii} - 1)(\hat{S}_{ii} - 9) = -(\hat{u}_i - 1)(\hat{u}_i - 9), \tag{12}$$

for $i = 1,\ldots,N$. Since $1 \le \hat{S}_{ii} \le 9$, we have $w_i \ge 0$. It follows that $\hat{\mathbf{U}} - \hat{\mathbf{u}}\hat{\mathbf{u}}^T = D(\mathbf{w}) \succeq \mathbf{0}$. Moreover, from (11)-(12), one can show that

$$\hat{U}_{ii} - 10\hat{u}_i + 9 = w_i + \hat{u}_i^2 - 10\hat{u}_i + 9 = 0$$

for all i. Hence, $(\hat{\mathbf{S}}, \hat{\mathbf{s}}, \hat{\mathbf{U}}, \hat{\mathbf{u}})$ is a feasible point of (9) yielding an objective value equal to $f_{\mathsf{BC-SDR}}$. This completes the proof of $f_{\mathsf{BC-SDR}} = f_{\mathsf{PI-SDR}}$. ■

Proposition 1 implies that the two SDR methods achieve the same approximation quality with respect to the true ML optimal value. Hence, if PI-SDR manages to achieve an optimal value equal to the true ML, so does BC-SDR and vice versa. In fact, simulation results in Section 6 will show that they yield very similar symbol error performance.

5. FAST INTERIOR-POINT ALGORITHM FOR BC-SDR

This section describes a specialized interior-point algorithm that is particularly suitable for BC-SDR.

5.1. Specialized Interior-Point Algorithm

Our development is based on the primal-dual path-following interior-point method by Helmberg *et. al* [12], which was found to be reliable in its applications in BPSK/QPSK SDR [2] and MPSK SDR [3]. The method, in its most general form, deals with any SDP that takes the form

$$\begin{array}{ll} \max & \mathrm{tr}(\mathbf{CX}) \\ \text{s.t.} & \mathbf{X} \succeq \mathbf{0}, \\ & \mathbf{a} - \mathcal{A}(\mathbf{X}) = \mathbf{0}, \quad \mathbf{b} - \mathcal{B}(\mathbf{X}) \succeq \mathbf{0} \end{array} \tag{13}$$

where $\mathbf{X} \in \mathbb{S}^n$, $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{b} \in \mathbb{R}^m$, $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^k$ and $\mathcal{B} : \mathbb{S}^n \to \mathbb{R}^m$ are linear functions. The interior-point method is iterative, requiring at most $\mathcal{O}(\sqrt{n})$ iterations to achieve a given solution accuracy. The bulk of its computational load lies in the complexity per iteration, which can be shown to be

$$\mathcal{O}((k+m)^2 n^2 + (k+m)n^3 + (k+m)^3 + n^3) \tag{14}$$

This complexity, however, refers only to cases where $\mathcal{A}$ and $\mathcal{B}$ are unstructured. For certain structured $\mathcal{A}$ and $\mathcal{B}$, it is possible to reduce the complexity; e.g., BPSK/QPSK and MPSK SDRs [2, 3].

We are interested in a special case of (13):

$$\begin{array}{ll} \max & \mathrm{tr}(\mathbf{CX}) \\ \text{s.t.} & \mathbf{X} \succeq \mathbf{0}, \\ & \mathbf{a} - \mathbf{A}d(\mathbf{X}) = \mathbf{0}, \quad \mathbf{b} - \mathbf{B}d(\mathbf{X}) \succeq \mathbf{0} \end{array} \tag{15}$$

where $\mathbf{X} \in \mathbb{S}^n$, $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{A} \in \mathbb{R}^{k \times n}$ and $\mathbf{B} \in \mathbb{R}^{m \times n}$. In essence, the equality and inequality constraints depend only on the diagonals of $\mathbf{X}$. By exploiting the special problem structures and by following the principle in [12], we build a specialized interior-point algorithm for (15). Since the derivations are quite cumbersome and long, we provide the algorithm in Table 1 without giving the mathematical details. The key difference of the new algorithm lies in Step 2, where we fully utilize the problem structures to make the search direction computations more effective. The complexity per iteration of the proposed algorithm is shown to be

$$\begin{aligned} &\mathcal{O}((k+m)^2 n + (k+m)n^2 + (k+m)^3 + n^3) \\ &= \mathcal{O}((\max\{k+m, n\})^3) \end{aligned}$$

which has a significant order reduction compared to its general-purpose counterpart; cf. (14).

5.2. Application to BC-SDR

Now let us turn our attention to the BC-SDR problem in (5). By Schur complement, (5) can be reformulated as [5]

$$\begin{array}{ll} \min\limits_{\mathbf{X} \in \mathbb{S}^{N+1}} & \mathrm{tr}\left(\begin{bmatrix} \mathbf{H}^T\mathbf{H} & -\mathbf{H}^T\mathbf{y} \\ -\mathbf{y}^T\mathbf{H} & \|\mathbf{y}\|_2^2 \end{bmatrix} \mathbf{X} \right) \\ \text{s.t.} & \mathbf{X} \succeq \mathbf{0}, \quad X_{N+1,N+1} = 1, \\ & 1 \le X_{ii} \le 9, \ i = 1, \dots, N \end{array} \tag{16}$$

where the relationship between (16) and (5) lies in:

$$\mathbf{X} = \begin{bmatrix} \mathbf{S} & \mathbf{s} \\ \mathbf{s}^T & 1 \end{bmatrix}.$$

Apparently, (16) can be expressed as (15) where $n = N+1$, $k = 1$, $m = 2N$, $\mathbf{A} = [\, 0, \dots, 0, 1 \,]$, $\mathbf{a} = 1$,

$$\mathbf{b} = \begin{bmatrix} -\mathbf{1}_N \\ 9\mathbf{1}_N \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -\mathbf{I}_N & \mathbf{0} \\ \mathbf{I}_N & \mathbf{0} \end{bmatrix}, \quad \mathbf{C} = -\begin{bmatrix} \mathbf{H}^T\mathbf{H} & -\mathbf{H}^T\mathbf{y} \\ -\mathbf{y}^T\mathbf{H} & \|\mathbf{y}\|_2^2 \end{bmatrix}$$

Table 1. Interior-point algorithm for (15). The notation $\circ$ stands for the Hadamard product, and $\mathbf{t}^{-1}$ elementwise inverse.

Given a primal-dual strictly feasible initial point $(\mathbf{X}, \boldsymbol{\nu}, \mathbf{t}, \mathbf{Z})$ and a solution accuracy $\epsilon > 0$.

Step 1. $\mu := [\mathrm{tr}(\mathbf{ZX}) + \mathbf{t}^T(\mathbf{b} - \mathbf{B}d(\mathbf{X}))]/[2(n+m)]$.

Step 2. Solve the linear equations

$$\mathbf{F} \begin{bmatrix} \Delta\boldsymbol{\nu} \\ \Delta\mathbf{t} \end{bmatrix} = \mathbf{g}$$

for search directions $(\Delta\boldsymbol{\nu}, \Delta\mathbf{t})$, where

$$\mathbf{F} := \begin{bmatrix} \mathbf{A}(\mathbf{Z}^{-1} \circ \mathbf{X})\mathbf{A}^T & \mathbf{A}(\mathbf{Z}^{-1} \circ \mathbf{X})\mathbf{B}^T \\ \mathbf{B}(\mathbf{Z}^{-1} \circ \mathbf{X})\mathbf{A}^T & \mathbf{B}(\mathbf{Z}^{-1} \circ \mathbf{X})\mathbf{B}^T + \mathbf{D} \end{bmatrix}$$

$$\mathbf{D} := D((\mathbf{b} - \mathbf{B}d(\mathbf{X})) \circ \mathbf{t}^{-1})$$

$$\mathbf{g} := \begin{bmatrix} \mu\mathbf{A}d(\mathbf{Z}^{-1}) - \mathbf{a} \\ \mu\mathbf{B}d(\mathbf{Z}^{-1}) - \mathbf{b} + \mu\mathbf{t}^{-1} \end{bmatrix}$$

Step 3. Compute the search directions

$$\Delta\mathbf{Z} := D(\mathbf{A}^T\Delta\boldsymbol{\nu} + \mathbf{B}^T\Delta\mathbf{t})$$

$$\Delta\mathbf{X} := \mu\mathbf{Z}^{-1} - \mathbf{X} - \mathbf{Z}^{-1}\Delta\mathbf{Z}\mathbf{X}$$

and symmetrize $\Delta\mathbf{X}$ by $\Delta\mathbf{X} := (\Delta\mathbf{X} + \Delta\mathbf{X}^T)/2$.

Step 4. Use line search to find a primal step-size $\alpha_p \in (0,1]$ such that $\mathbf{X} + \alpha_p\Delta\mathbf{X} \succ \mathbf{0}$ and $\mathbf{b} - \mathbf{B}d(\mathbf{X} + \alpha_p\Delta\mathbf{X}) \succ \mathbf{0}$.

Step 5. Use line search to find a dual step-size $\alpha_d \in (0,1]$ such that $\mathbf{Z} + \alpha_d\Delta\mathbf{Z} \succ \mathbf{0}$ and $\mathbf{t} + \alpha_d\Delta\mathbf{t} \succ \mathbf{0}$.

Step 6. Update $\mathbf{X} := \mathbf{X} + \alpha_p\Delta\mathbf{X}$, $\mathbf{Z} := \mathbf{Z} + \alpha_d\Delta\mathbf{Z}$, $\boldsymbol{\nu} := \boldsymbol{\nu} + \alpha_d\Delta\boldsymbol{\nu}$, and $\mathbf{t} := \mathbf{t} + \alpha_d\Delta\mathbf{t}$.

Step 7. If $\mathrm{tr}(\mathbf{ZX}) + \mathbf{t}^T(\mathbf{b} - \mathbf{B}d(\mathbf{X})) \le \epsilon$ (i.e., duality gap is less than ϵ) then terminate & output $(\mathbf{X}, \boldsymbol{\nu}, \mathbf{t}, \mathbf{Z})$; otherwise go to Step 1.

Using the specialized interior-point algorithm in Table 1, we can solve (16) with a total complexity of $\mathcal{O}(N^{3.5})$.

One implementation aspect worth mentioning is that the proposed algorithm requires a primal-dual strictly feasible point as an initialization (some general-purpose SDP solvers do not require initialization, but explicitly providing an initialization has the advantage of reducing the operational overheads). For BC-SDR where $\mathbf{C} \preceq \mathbf{0}$, the following initialization is appropriate:

$$\mathbf{X} = \begin{bmatrix} 5\mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \boldsymbol{\nu} = \gamma, \mathbf{t} = \begin{bmatrix} \gamma\mathbf{1}_N \\ 2\gamma\mathbf{1}_N \end{bmatrix}, \mathbf{Z} = D(\mathbf{A}^T\boldsymbol{\nu} + \mathbf{B}^T\mathbf{t}) - \mathbf{C}$$

where $\gamma = -\mathrm{tr}(\mathbf{C})/(N+1)$.

6. SIMULATION RESULTS AND CONCLUSIONS

In the first simulation example, we compare the symbol error rates (SERs) of the PI-SDR and BC-SDR detectors in an 8×8 MIMO system [i.e., $(\tilde{M}, \tilde{N}) = (8, 8)$]. The simulation settings follow those of a standard MIMO system [4, 5]. In the two SDR detectors, we employ the Gaussian randomization [4, 5] for solution approximation and the number of randomizations is 100. We also tested several other MIMO detectors, namely i) the zero-forcing (ZF) detector, ii) optimal sphere decoding [7], and iii) the lattice-reduction-aided ZF (LRA-ZF) detector which has been recently shown to have the full receive diversity [13]. The results are shown

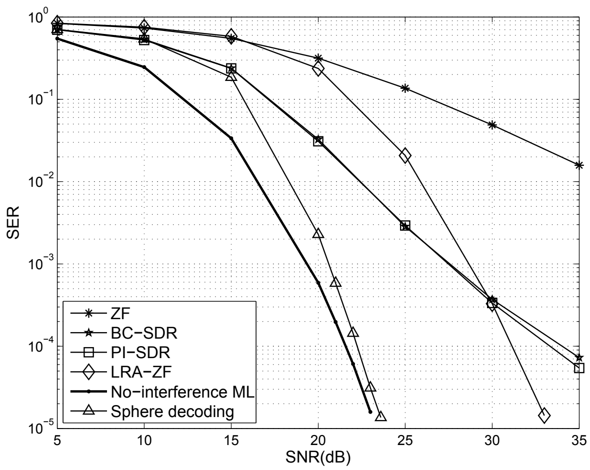

Fig. 1. Symbol error rates in an 8×8 16-QAM system.

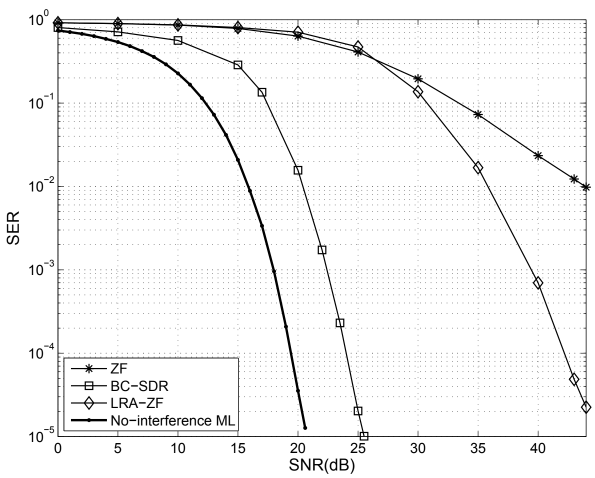

Fig. 2. Symbol error rates in a 40×40 16-QAM system.

in Fig. 1 and we have the following observations: First, the SERs of PI-SDR and BC-SDR are almost identical. This provides a strong support to the equivalent SDR relationship in Proposition 1. A subtle point to note is that the SERs of the two SDRs are not exactly equal, and that is due to the random natures of the Gaussian randomizations. Second, for SNRs greater than 30dB, the performance of the two SDR methods is surpassed by that of LRA-ZF.

In the second simulation example, we increase the problem size to $(\tilde{M}, \tilde{N}) = (40, 40)$. Sphere decoding was not tested because its complexity is too high in this example. We also omitted PI-SDR since its performance should be similar to that of BC-SDR. The results, shown in Fig. 2, illustrate that BC-SDR provides performance considerably better than LRA-ZF, in contrast to the small size problem in the last example. We spectulate that LRA-ZF has chance to outperform BC-SDR for sufficiently high SNRs, but the required SNR values may have to be very large.

In the third simulation example, we demonstrate the efficiency of our fast BC-SDR implementation (in Section 5) by comparing its computational time to its counterpart using the general-purpose solver SeDuMi. The SNR is fixed at 15dB. The test was conducted under MATLAB, using a desktop computer with dual 2.66GHz CPUs. The fast BC-SDR implementation was written in C mostly, with some minor operations using MATLAB. The results, shown in Table 2, indicate that the fast BC-SDR implementation ('Proposed' in the table) is many times faster than that using SeDuMi.

Table 2. Average computational time of the various methods.

	Time spent (in sec.)		
	$\tilde{N} = \tilde{M} = 5$	$\tilde{N} = \tilde{M} = 10$	$\tilde{N} = \tilde{M} = 20$
Proposed	0.0015	0.0051	0.0288
SeDuMi	0.0362	0.0709	0.2753

In conclusion, two new results for 16-QAM SDR detection have been presented. The first, which is theoretical, shows that PI-SDR and BC-SDR are equivalent in giving the same approximation quality. The second is practical where a specialized interior-point algorithm is developed for fast implementation of BC-SDR.

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