APPROXIMATION BOUND FOR SEMIDEFINITE RELAXATION BASED MULTICAST TRANSMIT BEAMFORMING

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ABSTRACT

The max-min-fair transmit beamforming problem in multigroup broadcasting has been shown to be NP-hard in general. Recently, a polynomial time approximation approach based on semidefinite relaxation (SDR) has been proposed [1]. It was found [1], through computer simulations, that this method is capable of giving a good approximate solution in polynomial time. This paper shows that the SDR based approach can guarantee as least an $\mathcal{O}(1/M)$ approximation quality, where M is the total number of receivers. The existence of such a data independent bound certifies the worst case approximation quality of the SDR algorithm for any problem instance and any number of transmit antennas.

Index Terms— Transmit beamforming, broadcasting, semidefinite relaxation, approximation bound.

1. INTRODUCTION

The multicast transmit beamforming problem has become a subject of great interest recently [1–5]. In contrast to traditional broadcasting methods which radiate signal power isotropically and separate groups using non-overlapping spectra, the multicast transmit beamforming uses the multiple antennas at the base station to form appropriate beampatterns for multiple receivers in different groups over a common frequency band. The beampatterns can be specially designed to reduce the cochannel interference between groups and to achieve desired quality of service at each receiver.

For the max-min-fair transmit beamformer [1, 3], the beampatterns are designed such that minimum signal to interference plus noise ratio (SINR) at receivers is maximized under a power constraint at the base station. In the single-group broadcasting scenario [5], the max-min-fair transmit beamforming problem has been shown to be NP-hard in general. Fortunately, it is possible to obtain a good approximate solution by a semidefinite relaxation (SDR) approach [5]. The SDR approach requires solving an associated semidefinite program followed by a simple randomization procedure to generate a feasible solution, all of which can be completed in polynomial time. It has been proved in [6] that, in the single-group case the SDR approach essentially guarantees a worst case $\mathcal{O}(1/m_1)$ approximation quality, where m_1 denotes the total number of receivers.

In the general multigroup scenario, it was found [1] through extensive computer simulations that the solution of the max-min-fair transmit beamforming problem can also be approximated well by a SDR based approach. This approach solves a generalized linear fractional program (in polynomial time) followed by a randomization/multigroup-power-control loop [1]. In this paper, we present an analysis which shows that the SDR based approach can provide at least $\mathcal{O}(1/\sum_{k=1}^{G} m_k)$ approximation quality, where G denotes the number of groups and m_k is the number of receivers in group k. Some simulation results are also presented to illustrate the empirical worst case and average approximation qualities of the SDR based approach.

2. MULTICAST TRANSMIT BEAMFORMING

Consider a scenario where a base station equipped with N transmit antennas broadcasts G ($G > 1$) independent data streams to M ($M = \sum_{k=1}^{G} m_k$) single-antenna receivers over a common frequency band. Each of the receivers belongs to one of the G groups, and the receivers in a group are interested in a common data stream. Let $s_k(t)$ and $\boldsymbol{w}_k \in \mathbb{C}^N$ denote the broadcasting data stream and the transmit weight vector for the kth group, respectively. The transmitted signal in the base station is given by $\sum_{k=1}^{G} \boldsymbol{w}_k^H s_k(t)$. Assume that $s_k(t)$, $k = 1, \ldots, G$, are statistically independent and are temporally white with zero mean and unit variance. Let $\boldsymbol{h}_{k,i} \in \mathbb{C}^N$ denote the random channel vector between the base station and the ith receiver in group k. The SINR of the ith receiver in group k is given by

$$\mathrm{SINR}_{k,i} = \frac{\boldsymbol{w}_k^H \mathbf{R}_{k,i} \boldsymbol{w}_k}{\sum_{j \neq k} \boldsymbol{w}_j^H \mathbf{R}_{k,i} \boldsymbol{w}_j + \sigma_{k,i}^2}$$

where $\mathbf{R}_{k,i} = \mathrm{E}\{\boldsymbol{h}_{k,i}\boldsymbol{h}_{k,i}^H\}$ is the channel correlation matrix and $\sigma_{k,i}^2$ is the noise variance.

The idea of transmit beamfoming is to design the weight vectors $\boldsymbol{w}_k$ such that each receiver can retrieve the signal of interest with desired quality of service. The quality of service is usually measured in terms of SINR. The max-min-fair transmit beamforming [1, 3] is one of the effective criteria to achieve this goal. It maximizes the minimum SINR among M receivers subject to the power constraint of $P > 0$ in the base station. Mathematically, it can be formulated as the following optimization problem

$$u^\star = \max_{\substack{\boldsymbol{w}_k \in \mathbb{C}^N \\ k=1,\ldots,G}} \; \min_{\substack{i=1,\ldots,m_k \\ k=1,\ldots,G}} \frac{\boldsymbol{w}_k^H \mathbf{R}_{k,i} \boldsymbol{w}_k}{\sum_{j \neq k} \boldsymbol{w}_j^H \mathbf{R}_{k,i} \boldsymbol{w}_j + \sigma_{k,i}^2} \qquad (1)$$

$$\text{s.t.} \; \sum_{k=1}^{G} \|\boldsymbol{w}_k\|^2 \leq P.$$

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It has been shown in [1] that problem (1) is NP-hard in general [1], therefore an approximation method for obtaining a good approximate solution in polynomial time is desired. In [1], Karipidis et al. proposed an approximation method based on SDR. To illustrate this, we define $\mathbf{W}_k = \boldsymbol{w}_k \boldsymbol{w}_k^H$ and rewrite problem (1) as

$$
\begin{aligned}
u^\star = \max_{\substack{\mathbf{W}_k \in \mathbb{C}^{N\times N}\\ k=1,\ldots,G}} \; & \min_{\substack{i=1,\ldots,m_k\\ k=1,\ldots,G}} \frac{\mathrm{tr}(\mathbf{R}_{k,i}\mathbf{W}_k)}{\sum_{j\neq k}\mathrm{tr}(\mathbf{R}_{k,i}\mathbf{W}_j)+\sigma_{k,i}^2} \\
\text{s.t.} \; & \sum_{k=1}^{G}\mathrm{tr}(\mathbf{W}_k) \le P, \\
& \mathbf{W}_k \succeq \mathbf{0}, \;\; \mathrm{rank}(\mathbf{W}_k)=1, \; k=1,\ldots,G,
\end{aligned} \tag{2}
$$

where $\mathrm{tr}(\cdot)$ stands for the trace of a matrix. By dropping the only non-convex constraint $\mathrm{rank}(\mathbf{W}_k)=1$, we obtain the following relaxation counterpart of problem (1)

$$
\begin{aligned}
v^\star = \max_{\substack{\mathbf{W}_k \in \mathbb{C}^{N\times N}\\ k=1,\ldots,G}} \; & \min_{\substack{i\in\mathcal{G}_k\\ k=1,\ldots,G}} \frac{\mathrm{tr}(\mathbf{R}_{k,i}\mathbf{W}_k)}{\sum_{j\neq k}\mathrm{tr}(\mathbf{R}_{k,i}\mathbf{W}_j)+\sigma_{k,i}^2} \\
\text{s.t.} \; & \sum_{k=1}^{G}\mathrm{tr}(\mathbf{W}_k) \le P, \\
& \mathbf{W}_k \succeq \mathbf{0}, \; k=1,\ldots,G.
\end{aligned} \tag{3}
$$

Note that $v^\star \ge u^\star$ since the feasible set of problem (2) is a subset of that of problem (3). Instead of being NP-hard as problem (1), problem (3) is a generalized convex linear fractional program and can be solved by the bisection algorithm [8] in polynomial time. The SDR based approximation method proposed in [1] is to solve problem (3) in the first stage. In the second stage, based on the solution of (3) a randomization/multigroup-power-control loop is applied to obtain an approximate solution of problem (1). Please refer to [1] for the details.

3. ANALYSIS OF APPROXIMATION BOUND

In this section, we show that the SDR based approximation approach mentioned in Section 2 has at least $\mathcal{O}(1/M)$ approximation quality. To this end, let us consider the following problem which is closely related to problem (1)

$$
\begin{aligned}
u^\star = \max_{\boldsymbol{w}\in\mathbb{C}^n} \min_{j=1,\ldots,M} & \frac{\boldsymbol{w}^H\mathbf{A}_j\boldsymbol{w}}{\boldsymbol{w}^H\mathbf{B}_j\boldsymbol{w}+1} \\
\text{s.t. } & \boldsymbol{w}^H\boldsymbol{w} \le P.
\end{aligned} \tag{4}
$$

By defining $\boldsymbol{w} = [\boldsymbol{w}_1^T,\ldots,\boldsymbol{w}_G^T]^T \in \mathbb{C}^n$ where $n = GN$, one can reformulate problem (1) into the same form as problem (4) with matrices $\mathbf{A}_j$ and $\mathbf{B}_j$ being block diagonal matrices defined by channel correlation matrices $\mathbf{R}_{k,i}$ and noise variance $\sigma_{k,i}^2$. Thus problem (4) serves as a generalization of problem (1), and the approximation bound for (4) can directly apply to problem (1). It follows from (2) and (3) that the relaxation counterpart of problem (4) is given by

$$
\begin{aligned}
v^\star = \max_{\mathbf{W}\in\mathbb{C}^{n\times n}} \min_{j=1,\ldots,M} & \frac{\mathrm{tr}(\mathbf{A}_j\mathbf{W})}{\mathrm{tr}(\mathbf{B}_j\mathbf{W})+1} \\
\text{s.t. } & \mathrm{tr}(\mathbf{W}) \le P, \\
& \mathbf{W} \succeq \mathbf{0}.
\end{aligned} \tag{5}
$$

The main theoretical result of this paper is presented in the following theorem.

[1] Two special scenarios for which problem (1) can be solved in polynomial time can be found in [7] and [2].

Theorem 1 *For problem (4) and its relaxation problem (5), the ratio $v^\star/u^\star = 1$ for $M \le 3$, and for $M > 3$*

$$1 \le \frac{v^\star}{u^\star} \le 30M. \tag{6}$$

Besides, for $M > 3$ consider the randomization procedure:

(S1) *Solve (5) using bisection algorithm and let $\mathbf{W}^\star$ be an optimal solution. Let $L > 0$ be an integer.*

(S2) *For $\ell = 1,2,\ldots,L$, generate a random vector $\boldsymbol{\xi} \in \mathbb{C}^n$ from the complex Gaussian distribution $\mathcal{N}_c(\mathbf{0},\mathbf{W}^\star)$, and let*

$$\boldsymbol{w}^{(\ell)} = \sqrt{P}\boldsymbol{\xi}/\|\boldsymbol{\xi}\|$$

and

$$u^{(\ell)} = \min_{j=1,\ldots,M} \frac{(\boldsymbol{w}^{(\ell)})^H\mathbf{A}_j\boldsymbol{w}^{(\ell)}}{(\boldsymbol{w}^{(\ell)})^H\mathbf{B}_j\boldsymbol{w}^{(\ell)}+1}. \tag{7}$$

(S3) *Suppose*

$$\ell^\star = \arg \max_{\ell=1,2,\ldots,L} \{u^{(\ell)}\}.$$

Then select $\boldsymbol{w}^{(\ell^\star)}$ as the approximate optimum solution of (4), and let $\hat{u}^\star = u^{(\ell^\star)}$.

Then

$$\frac{u^\star}{30M} \le \hat{u}^\star \le u^\star \tag{8}$$

with probability at least $1-(0.9393)^L$.

Note that (8) implies that the SDR based method can achieve at least $1/(30M)$ approximation quality in randomized polynomial time.

3.1. Proof of Theorem 1

The following lemmas are developed for the proof of Theorem 1:

Lemma 1 *The rank of the optimum solution $\mathbf{W}^\star$ of problem (5) is upper bounded by $\sqrt{M}$.*

Proof: For the function $f(x,y) = \frac{x}{y+1}$, $x > 0$ and $y \ge 0$, $f(\beta x, \beta y) > f(x,y)$ for any $\beta > 1$. Consider the optimization problem

$$
\begin{aligned}
\mathbf{X}^\star = \arg \min_{\mathbf{X}\in\mathbb{C}^{n\times n}} & \; \mathrm{tr}(\mathbf{X}) \\
\text{s.t. } & \frac{\mathrm{tr}(\mathbf{A}_j\mathbf{X})}{\mathrm{tr}(\mathbf{B}_j\mathbf{X})+1} \ge v^\star, \; j=1,\ldots,M, \\
& \mathbf{X} \succeq \mathbf{0}.
\end{aligned} \tag{9}
$$

It can be readily checked that $\mathbf{W}^\star$ is feasible for problem (9), and the power constraint in problem (5) would be active, i.e., $\mathrm{tr}(\mathbf{W}^\star) = P$. Hence $\mathrm{tr}(\mathbf{X}^\star) \le \mathrm{tr}(\mathbf{W}^\star)$. Suppose that $\mathrm{tr}(\mathbf{X}^\star) < \mathrm{tr}(\mathbf{W}^\star) = P$. One can scale $\mathbf{X}^\star$ to $\bar{\mathbf{X}}^\star = \beta\mathbf{X}^\star$ where $\beta = P/\mathrm{tr}(\mathbf{X}^\star) > 1$, and obtain

$$\mathrm{tr}(\bar{\mathbf{X}}^\star) = P, \quad \min_{j=1,\ldots,M} \frac{\mathrm{tr}(\mathbf{A}_j\bar{\mathbf{X}}^\star)}{\mathrm{tr}(\mathbf{B}_j\bar{\mathbf{X}}^\star)+1} > v^\star,$$

which contradicts the optimality of $\mathbf{W}^\star$ for (5). Thus $\mathrm{tr}(\mathbf{X}^\star) = \mathrm{tr}(\mathbf{W}^\star)$, i.e., $\mathbf{W}^\star$ is also an optimum solution of (9). Since problem (9) is a semidefinite program, it has been shown [9] that

$$\mathrm{rank}(\mathbf{W}^\star) \le \sqrt{M}.$$

■

Lemma 2 *Let* $\mathbf{A} \in \mathbb{C}^{n\times n}$ *and* $\mathbf{B} \in \mathbb{C}^{n\times n}$ *be two Hermitian positive semidefinite matrices (*$\mathbf{A} \succeq \mathbf{0}, \mathbf{B} \succeq \mathbf{0}, \mathbf{B} \neq \mathbf{0}$*), and* $\boldsymbol{\xi} \in \mathbb{C}^n$ *be a random vector with complex Gaussian distribution* $\mathcal{N}_c(\mathbf{0}, \mathbf{W}^\star)$*. Then*

$$\Pr\left(\frac{\boldsymbol{\xi}^H\mathbf{A}\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}\boldsymbol{\xi}+1} < \gamma\frac{\mathrm{E}(\boldsymbol{\xi}^H\mathbf{A}\boldsymbol{\xi})}{\mathrm{E}(\boldsymbol{\xi}^H\mathbf{B}\boldsymbol{\xi})+1}\right) \leq \max\left\{\frac{3\gamma}{\alpha-2\gamma}, \left(\frac{5\gamma}{\frac{1-\alpha}{\bar{r}-1}-3\gamma}\right)^2\right\}$$

where $\bar{r} = \min\{\mathrm{rank}(\mathbf{A}), \mathrm{rank}(\mathbf{W}^\star)\}$, $0 \leq \gamma < \min\{\frac{\alpha}{2}, \frac{1}{3}\frac{1-\alpha}{\bar{r}-1}\}$*, and* $0 < \alpha < 1$.

Due to space limit, the proof of Lemma 2 (which will be reported in [10]) is omitted here. With Lemmas 1 to 2, we can prove Theorem 1.

Proof of Theorem 1: By Lemma 1, for $M \leq 3$ there exists a solution of problem (5) with $\mathrm{rank}(\mathbf{W}^\star) = 1$. Hence for $M \leq 3$, $v^\star = u^\star$. This rank-1 solution, which is feasible to (4) and has an objective value $\hat{u}^\star$ equal to $u^\star$, can always be obtained via a matrix decomposition procedure [9]. Therefore, $\gamma_{\mathrm{sdr}} = 1$ for $M \leq 3$. To obtain (6) for $M > 3$, we consider to prove that

$$\Pr\left(\min_{j=1,\ldots,M}\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1} \geq \gamma\, v^\star,\ \boldsymbol{\xi}^H\boldsymbol{\xi} \leq \mu P\right) > 0 \qquad (10)$$

for $\gamma = 1/(16M)$ and $\mu = 15/8$, where $\boldsymbol{\xi} \in \mathbb{C}^n$ is a random vector with complex Gaussian distribution $\mathcal{N}_c(\mathbf{0}, \mathbf{W}^\star)$. If (10) is true, then there exists a realization of $\boldsymbol{\xi}$ which satisfies $\boldsymbol{\xi}^H\boldsymbol{\xi} \leq (15/8)P$ and

$$\min_{j=1,\ldots,M}\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1} \geq \left(\frac{1}{16M}\right)v^\star.$$

Let $\bar{\boldsymbol{\xi}} = \sqrt{8/15}\boldsymbol{\xi}$ which then is feasible for problem (4) (i.e., $\bar{\boldsymbol{\xi}}^H\bar{\boldsymbol{\xi}} \leq P$) and satisfies

$$\left(\frac{1}{30M}\right)v^\star \leq \min_{j=1,\ldots,M}\frac{\bar{\boldsymbol{\xi}}^H\mathbf{A}_j\bar{\boldsymbol{\xi}}}{\bar{\boldsymbol{\xi}}^H\mathbf{B}_j\bar{\boldsymbol{\xi}}+1} \leq u^\star, \qquad (11)$$

which is part of (6).

We now prove (10). Note that the left hand side (L.H.S.) of (10) can be lower bounded as follows

$$\begin{aligned}
&\Pr\left(\min_{j=1,\ldots,M}\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1} \geq \gamma v^\star,\ \boldsymbol{\xi}^H\boldsymbol{\xi} \leq \mu P\right)\\
&\geq 1 - \sum_{j=1}^{M}\Pr\left(\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1} < \gamma v^\star\right) - \Pr\left(\boldsymbol{\xi}^H\boldsymbol{\xi} > \mu P\right)\\
&\geq 1 - \sum_{j=1}^{M}\Pr\left(\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1} < \gamma\frac{\mathrm{tr}(\mathbf{A}_j\mathbf{W}^\star)}{\mathrm{tr}(\mathbf{B}_j\mathbf{W}^\star)+1}\right)\\
&\qquad - \Pr\left(\boldsymbol{\xi}^H\boldsymbol{\xi} > \mu\,\mathrm{tr}(\mathbf{W}^\star)\right) \quad \text{(by (5))}\\
&= 1 - \sum_{j=1}^{M}\Pr\left(\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1} < \gamma\frac{\mathrm{E}\{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}\}}{\mathrm{E}\{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}\}+1}\right)\\
&\qquad - \Pr\left(\boldsymbol{\xi}^H\boldsymbol{\xi} > \mu\mathrm{E}\{\boldsymbol{\xi}^H\boldsymbol{\xi}\}\right) \quad (\text{since } \boldsymbol{\xi} \sim \mathcal{N}_c(\mathbf{0}, \mathbf{W}^\star))\\
&\geq 1 - \sum_{j=1}^{M}\Pr\left(\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1} < \gamma\frac{\mathrm{E}\{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}\}}{\mathrm{E}\{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}\}+1}\right) - \frac{1}{\mu} \qquad (12)\\
&\geq \frac{\mu-1}{\mu} - \sum_{j=1}^{M}\max\left\{\frac{3\gamma}{\alpha-2\gamma}, \left(\frac{5\gamma}{\frac{1-\alpha}{\bar{r}-1}-3\gamma}\right)^2\right\}, \qquad (13)
\end{aligned}$$

where (12) is by Markov inequality, and the last step (13) is due to Lemma 2. Because $\bar{r} \leq \sqrt{M}$ by Lemma 1, by choosing $\alpha = 0.4932$ and $\gamma = 1/(16M)$ one can show that for $M > 3$,

$$\frac{3\gamma}{\alpha-2\gamma} \geq \left(\frac{5\gamma}{\frac{1-\alpha}{\bar{r}-1}-3\gamma}\right)^2. \qquad (14)$$

Hence, for $M > 3$ and $\mu = 15/8$,

$$\begin{aligned}
&\Pr\left(\min_{j=1,\ldots,M}\frac{\boldsymbol{\xi}^H\mathbf{A}_j\boldsymbol{\xi}}{\boldsymbol{\xi}^H\mathbf{B}_j\boldsymbol{\xi}+1} \geq \gamma v^\star,\ \boldsymbol{\xi}^H\boldsymbol{\xi} \leq \mu P\right)\\
&\geq \frac{7}{15} - M\frac{3\gamma}{\alpha-2\gamma} = \frac{7}{15} - \frac{3}{7.8912-2/M} > 0.0607, \qquad (15)
\end{aligned}$$

which establishes (10). This further implies that (11) holds.

To complete the proof, let $\hat{u}^\star$ be generated by the SDR procedure for solving (4). Then, for each ℓ, it follows from (7), (11) and (15) that $\boldsymbol{w}^{(\ell)} = \sqrt{P}\boldsymbol{\xi}/\|\boldsymbol{\xi}\|$ satisfies

$$\left(\frac{1}{30M}\right)u^\star \leq \min_{j=1,\ldots,M}\frac{(\boldsymbol{w}^{(\ell)})^H\mathbf{A}_j\boldsymbol{w}^{(\ell)}}{(\boldsymbol{w}^{(\ell)})^H\mathbf{B}_j\boldsymbol{w}^{(\ell)}+1} = u^{(\ell)}, \qquad (16)$$

with probability at least 0.0607. If one generates L independent realizations of $\boldsymbol{\xi}$ from the distribution $\mathcal{N}_c(\mathbf{0}, \mathbf{W}^\star)$, then it is at least with probability[2] $1-(1-0.0607)^L$ to obtain one $\boldsymbol{\xi}$ which can achieve the approximation quality in (16). Since $\hat{u}^\star = \max\{u^{(1)}, \ldots, u^{(L)}\}$, it follows that (8) holds with probability at least $1-(0.9393)^L$. Theorem 1 is proved. ■

4. SIMULATIONS AND DISCUSSIONS

In Section 3, we have presented a worst case analysis for the SDR based approximation method [1] for the multicast max-min-fair transmit beamforming problem. We have shown that the SDR based method can guarantee at least an $\mathcal{O}(1/M)$ approximation quality for any problem instance (e.g., any channel correlation matrix $\mathbf{R}_{k,i}$ and noise variance $\sigma_{k,i}^2$) and any number of transmit antennas N.

While the worst case approximation bounds in (6) and (8) have their own theoretical significance, the empirical approximation quality may be of great interest in practical applications. Here let us present the empirical approximation qualities for problems (4) and (5) from 1000 randomly generated problem instances. For each problem instance, the positive semidefinite matrices $\mathbf{A}_j$ and $\mathbf{B}_j$ were generated as follows [11]. For full rank $\mathbf{A}_j$, we set

$$\mathbf{A}_j = \mathbf{Q}^H * \mathrm{diag}\{\mathrm{randn}(n,1)\} * \mathbf{Q} \qquad (17)$$

where "$*$" and "randn" are Matlab notations, and $\mathbf{Q} \in \mathbb{C}^{n\times n}$ is a unitary matrix obtained by QR factorization of a randomly generated $n\times n$ complex matrix. For rank-1 $\mathbf{A}_j$, we set

$$\mathbf{A}_j = \mathbf{Q}^H * \mathrm{diag}\{\mathrm{randn}(1), \mathbf{0}_{n-1}^T\} * \mathbf{Q}, \qquad (18)$$

where $\mathbf{0}_{n-1}$ is the $(n-1)\times 1$ zero vector. Matrices $\mathbf{B}_j$ were generated in the same way as full rank matrices $\mathbf{A}_j$. The problem (5) was solved by the bisection algorithm [8] wherein SeDuMi [12] was employed to handle the associated feasibility problems. The randomization procedure was performed as in Theorem 1 with $L = 500$. Because the empirical quality bound $v^\star/\hat{u}^\star$ is greater than the true ratio $v^\star/u^\star$, the former was used to approximate the latter.

[2] For $L = 50$, this probability is 0.9563.

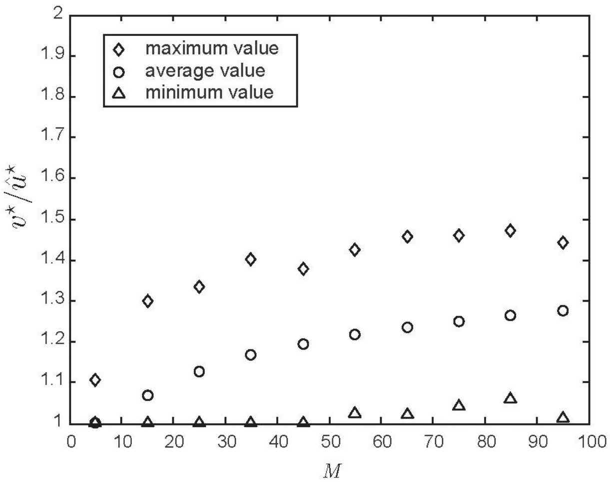

Figure 1. Empirical approximation qualities for $n = 10$, $P = 10$ and full rank $\mathbf{A}_j$.

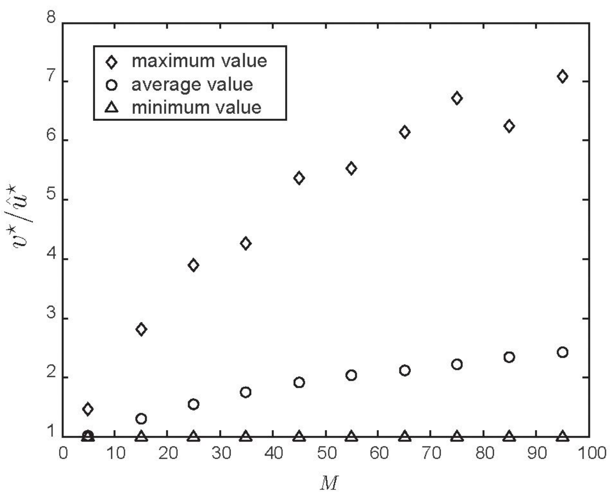

Figure 2. Empirical approximation qualities for $n = 10$, $P = 10$ and rank one $\mathbf{A}_j$.

Figures 1 and 2 present the empirical qualities for $n = 10$ and $P = 10$ when $\mathbf{A}_j$ are full rank and are rank one, respectively. The symbols "$\Diamond$" ("$\triangle$") denote the maximum (minimum) value of $v^\star/\hat{u}^\star$ for 1000 problem instances, and the symbols "$\circ$" represent the average value. One can see from these figures that in the average sense the SDR based approximation method provides very good approximation quality ($v^\star/u^\star \leq v^\star/\hat{u}^\star < 3$), and the bounds get larger when M increases. From Fig. 2, one can also observe that the symbol "$\Diamond$" of maximum value increases with M roughly in a linear manner. In fact, one can show [10] that the presented bound in (6) can be tight (up to a constant scalar) in a special problem scenario where all $\mathbf{A}_j$ are rank one. Figure 3 shows the histogram of empirical qualities for $n = 10$, $P = 10$ and $M = 45$. From Figures 1 to 3, one can see that the approximation quality when $\mathbf{A}_j$ are full rank is much better than that when $\mathbf{A}_j$ are rank one.

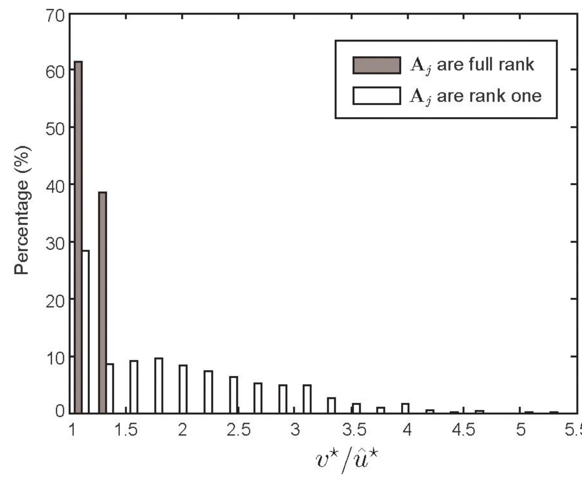

Figure 3. Histogram of empirical approximation qualities for $n = 10$, $P = 10$ and $M = 45$.

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