A SEMIDEFINITE RELAXATION BASED CONSERVATIVE APPROACH TO ROBUST TRANSMIT BEAMFORMING WITH PROBABILISTIC SINR CONSTRAINTS

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ABSTRACT

This paper considers a multiuser transmit beamforming design in the presence of Gaussian distributed channel errors at the transmitter. We study a stochastic robust design problem that minimizes the transmission power subject to probabilistic signal-to-interference-plus-noise ratio (SINR) constraints on the receivers. While the probabilistically-constrained design problem is difficult to solve, we present a conservative approximation formulation that guarantees the satisfaction of the probabilistic SINR requirements. The proposed conservative formulation is reminiscent of the worst-case robust design problem with spherically bounded channel errors, and can be efficiently handled by semidefinite relaxation. The presented simulation results show that the proposed approach outperforms the existing methods and requires less transmission powers under the same SINR requirement.

1. INTRODUCTION

Multiuser transmit beamforming designs that take into account channel state information (CSI) errors at the transmitter, or so called robust transmit beamforming, have drawn significant attention recently [1–6]. Depending on the CSI error model, robust transmit beamforming can be divided into two classes: The worst-case robust design and the stochastic robust design. The worst-case robust design assumes a bounded channel error model, and the beamforming vectors are designed such that the receivers' quality-of-service (QoS) requirements are fulfilled for all channels satisfying this model; e.g., see the works in [2] where the symbol mean squared error is used as the receiver's QoS measure, and also in [3] for signal-to-interference-plus-noise ratio (SINR) based QoS measure. In the stochastic robust design, the channel error is assumed to be random, following certain statistical distribution, and the beamforming vectors are designed such that the QoS requirement are met with a high probability. See [4–6] for the related works.

In this paper, we are interested in the stochastic robust transmit beamforming design. Specifically, we assume that the channel errors are independent and identically distributed (i.i.d.) complex Gaussian random vectors, and consider the design formulation in [4] where the transmission power is minimized subject to probabilistic SINR constraints. The probabilistic SINR constraints guarantee that the beamforming design provides each receiver with a preset, often high, probability that the desired SINR specification is satisfied (i.e., a high SINR satisfaction probability). The associated design problem is difficult because the probabilistic SINR constraints in general do not posses any tractable form. To efficiently handle this problem, conservative approaches, which can yield approximate solutions satisfying the probabilistic SINR requirements, have been proposed. For example, the conservative approaches presented in [4] can obtain such approximate solutions by simply solving a convex semidefinite program (SDP).

In this paper, we present a new conservative approach to approximating the probabilistically-constrained robust design problem. The idea of the proposed approach is to bypass the difficult probabilistic SINR constraints and replace them by worst-case SINR constraints with spherically bounded channel errors. We show that the spherically-bounded worst-case SINR constraints guarantee the satisfaction of the original probabilistic SINR constraints in the sense that the former constraints correspond to a problem feasible set that is subsumed by the latter constraints. The resultant conservative formulation resembles the worst-case robust transmit beamforming problem considered in [3,7], and hence can be conveniently handled by a semedefinite relaxation (SDR) based method [7]. Moreover, we illustrate that the level of conservatism of the proposed formulation can be reduced by a bisection methodology. Simulation results to be presented will demonstrate that the proposed conservative approach outperforms the methods in [4], and meanwhile draws less transmission powers under the same SINR requirement.

2. SIGNAL MODEL AND PROBLEM STATEMENT

2.1 Signal Model

We consider a multiuser wireless system that consists of a multiple-antenna transmitter and K single-antenna receivers. We assume that the transmitter is equipped with N_t antennas, and wants to transmit K independent data streams, denoted by $s_i(t), i = 1, \ldots, K$, to the K respective receivers using transmit beamforming [1]. The transmit signal is given by $\sum_{i=1}^{K} \boldsymbol{w}_i s_i(t)$, where $\boldsymbol{w}_i \in \mathbb{C}^{N_t}$ is the beamforming vector for $s_i(t)$. The received signal at receiver i can be expressed as

$$y_i(t) = \mathbf{h}_i^H \left(\sum_{i=1}^{K} \boldsymbol{w}_i s_i(t) \right) + n_i(t), \tag{1}$$

where $\mathbf{h}_i \in \mathbb{C}^{N_t}$ is the channel vector of receiver i, and $n_i(t)$ is the additive noise, with zero mean and variance $\sigma_i^2 > 0$.

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Assume that $\mathrm{E}[|s_i(t)|^2] = 1$ for all $i = 1, \ldots, K$. The SINR of receiver i can be obtained from (1) as

$$\mathrm{SINR}_i = \frac{\mathrm{E}[|\mathbf{h}_i^H \boldsymbol{w}_i s_i(t)|^2]}{\sum_{k\neq i}^K \mathrm{E}[|\mathbf{h}_i^H \boldsymbol{w}_k s_k(t)|^2] + \sigma_i^2} = \frac{|\mathbf{h}_i^H \boldsymbol{w}_i|^2}{\sum_{k\neq i}^K |\mathbf{h}_i^H \boldsymbol{w}_k|^2 + \sigma_i^2}.$$

Since the SINR is directly related to system performance such as bit error rate, it is commonly used as the receivers' QoS measure. The goal of the transmitter is to design the beamforming vectors $\{\boldsymbol{w}_i\}_{i=1}^K$ such that each of the receivers can meet a desired SINR requirement as

$$\mathrm{SINR}_i \geq \gamma_i, \ i = 1, \ldots, K, \tag{2}$$

where $\gamma_i > 0$ is the target SINR value for receiver i. To this end, the transmitter requires the receivers' CSI, i.e., $\{\mathbf{h}_i\}_{i=1}^K$. Suppose that the transmitter perfectly knows $\{\mathbf{h}_i\}_{i=1}^K$. A set of optimum beamforming vectors can be obtained by solving the following optimization problem

$$\begin{aligned} \min_{\substack{\boldsymbol{w}_i \in \mathbb{C}^{N_t}, \\ i=1,\ldots,K}} \ & \sum_{i=1}^K ||\boldsymbol{w}_i||^2 \\ \text{s.t.} \ & \frac{|\mathbf{h}_i^H \boldsymbol{w}_i|^2}{\sum_{k\neq i}^K |\mathbf{h}_i^H \boldsymbol{w}_k|^2 + \sigma_i^2} \geq \gamma_i, \quad i = 1, \ldots, K, \end{aligned} \tag{3}$$

where $||\cdot||$ denotes the vector 2-norm. As seen in (3), our task is to find a most power-efficient design that guarantees the desired SINR specifications in (2). It has been shown in [1] and [8] that problem (3) can be reformulated as a convex second-order cone program (SOCP). Therefore, efficient solvers, such as `CVX` [9], can be employed to solve problem (3).

2.2 Probabilistically-Constrained Robust Design

In wireless systems, CSI at the transmitter is either estimated via uplink training (e.g., in time division duplex systems) or acquired through receivers' feedback (e.g., in frequency division duplex systems). Hence it is inevitable to have CSI errors in practice due to finite-length training data or due to limited quantization feedback [1]. Let $\bar{\mathbf{h}}_i$, $i = 1, \ldots, K$, denote the obtained channel estimates at the transmitter. The true channel vectors $\{\mathbf{h}_i\}_{i=1}^K$ can be modeled as

$$\mathbf{h}_i = \bar{\mathbf{h}}_i + \mathbf{e}_i, \ i = 1, \ldots, K, \tag{4}$$

where $\mathbf{e}_i \in \mathbb{C}^{N_t}$ represents the channel error. With the imperfect CSI $\{\bar{\mathbf{h}}_i\}_{i=1}^K$, the standard transmit beamforming design in (3) may no longer guarantee the desired SINR specifications [in (2)] all the time, and therefore SINR outage may occur. This issue motivates the investigation of robust transmit beamforming design.

In this paper, we consider the stochastic robust transmit beamforming design. Specifically, we model each $\mathbf{e}_i$ as an i.i.d. complex Gaussian random vector, with zero mean and covariance matrix $\epsilon_i^2 \mathbf{I}_{N_t}$, i.e., $\mathbf{e}_i \sim \mathcal{CN}(\mathbf{0}, \epsilon_i^2 \mathbf{I}_{N_t})$. This model is particularly suitable for errors owing to imperfect channel estimation. Under this stochastic model, it is desirable to have a beamforming design that is able to guarantee the SINR specifications in (2) with a high satisfaction probability, or, equivalently, a low SINR outage probability. Let $\rho_i \in (0, 1]$ denotes the maximum tolerable SINR outage probability for receiver i. We consider the following probabilistically-constrained design formulation [4]

$$\min_{\substack{\boldsymbol{w}_i \in \mathbb{C}^{N_t}, \\ i=1,\ldots,K}} \ \sum_{i=1}^K ||\boldsymbol{w}_i||^2 \tag{5a}$$

$$\text{s.t.} \ \Pr\left(\frac{|(\bar{\mathbf{h}}_i + \mathbf{e}_i)^H \boldsymbol{w}_i|^2}{\sum_{k\neq i}^K |(\bar{\mathbf{h}}_i + \mathbf{e}_i)^H \boldsymbol{w}_k|^2 + \sigma_i^2} \geq \gamma_i \right) \geq 1 - \rho_i, \ i = 1, \ldots, K. \tag{5b}$$

Problem (5) aims to find a most power-efficient beamforming design that guarantees receiver i a $(1-\rho_i)$ SINR satisfaction probability, for $i = 1, \ldots, K$. Solving problem (5) is a challenging task because the probabilistic SINR constraints in (5b) are intractable and are not convex in general. In the next section, we present a conservative approach to approximating problem (5).

3. PROPOSED CONSERVATIVE APPROACH

In this section, the proposed conservative approach is presented. In the first subsection, a conservative formulation of problem (5) is given. In the second subsection, an SDR based method is presented to handle the proposed conservative formulation. A bisection technique that can further enhance the proposed conservative approach is presented in the last subsection.

3.1 Proposed Conservative Formulation

To present the proposed approach, let us write

$$\mathbf{e}_i = \epsilon_i \mathbf{v}_i \tag{6}$$

where $\mathbf{v}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{N_t})$, and rewrite the probabilistic SINR constraints in (5b) as

$$\Pr\left(\frac{|(\bar{\mathbf{h}}_i + \epsilon_i \mathbf{v}_i)^H \boldsymbol{w}_i|^2}{\sum_{k\neq i}^K |(\bar{\mathbf{h}}_i + \epsilon_i \mathbf{v}_i)^H \boldsymbol{w}_k|^2 + \sigma_i^2} \geq \gamma_i \right) \geq 1 - \rho_i \tag{7}$$

for $i = 1, \ldots, K$. The proposed conservation approach is based on the following simple lemma:

Lemma 1 Let $\mathbf{v} \in \mathbb{C}^{N_t}$ be a continuous random vector following certain statistical distribution and let $G(\mathbf{v}) : \mathbb{C}^{N_t} \to \mathbb{R}$ be a function of $\mathbf{v}$. Let $r > 0$ be the radius of the ball $\{\mathbf{v} | \ \|\mathbf{v}\|^2 \leq r^2\}$ such that $\Pr(\|\mathbf{v}\|^2 \leq r^2) \geq 1 - \rho$ where $\rho \in (0, 1]$. Then

$$G(\mathbf{v}) \geq 0 \ \ \forall \ \|\mathbf{v}\|^2 \leq r^2 \tag{8}$$

implies $\Pr(G(\mathbf{v}) \geq 0) \geq 1 - \rho$.

Proof: Let $f(\mathbf{v})$ denote the probability density function of $\mathbf{v}$. Suppose that (8) holds. We have that

$$\begin{aligned} \Pr(G(\mathbf{v}) \geq 0) &= \int_{G(\mathbf{v})\geq 0} f(\mathbf{v}) \ d\mathbf{v} \geq \int_{\|\mathbf{v}\|^2 \leq r^2} f(\mathbf{v}) \ d\mathbf{v} \\ &= \Pr\left(\|\mathbf{v}\|^2 \leq r^2\right) \geq 1 - \rho, \end{aligned} \tag{9}$$

where the first inequality follows from (8). The lemma then is proved. ■

By applying Lemma 1 to (7), we can see that (7) is satisfied whenever

$$\frac{|(\bar{\mathbf{h}}_i + \epsilon_i \mathbf{v}_i)^H \boldsymbol{w}_i|^2}{\sum_{k\neq i}^K |(\bar{\mathbf{h}}_i + \epsilon_i \mathbf{v}_i)^H \boldsymbol{w}_k|^2 + \sigma_i^2} \geq \gamma_i \ \ \forall \ \|\mathbf{v}_i\|^2 \leq r_i^2, \tag{10}$$

where $r_i > 0$ is the radius of the ball $\{\mathbf{v}_i | \ \|\mathbf{v}_i\|^2 \le r_i^2\}$ such that $\Pr(\|\mathbf{v}_i\|^2 \le r_i^2) \ge 1-\rho_i$. We should mention that Lemma 1 does not impose any specific statistical distribution on $\mathbf{v}_i$. However, for the case of $\mathbf{v}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_{N_t})$ which is of the main interest in this paper, $2\|\mathbf{v}_i\|^2$ is a Chi-square random variable with $2N_t$ degrees of freedom. Let ICDF$(\cdot)$ be the inverse cumulative distribution function of $2\|\mathbf{v}_i\|^2$. Then one can show that the choice of the following radius

$$r_i = \sqrt{\frac{\text{ICDF}(1-\rho_i)}{2}} \tag{11}$$

is sufficient to guarantee $\Pr(\|\mathbf{v}_i\|^2 \le r_i^2) \ge 1-\rho_i$ for $i = 1,\ldots,K$. Therefore, with the radii $\{r_i\}_{i=1}^K$ in (11), and by Lemma 1, the following problem

$$\begin{aligned} \min_{\substack{\boldsymbol{w}_i \in \mathbb{C}^{N_t},\\ i=1,\ldots,K}} \ & \sum_{i=1}^K ||\boldsymbol{w}_i||^2 \qquad (12)\\ \text{s.t.} \ & \frac{|(\bar{\mathbf{h}}_i + \epsilon_i \mathbf{v}_i)^H \boldsymbol{w}_i|^2}{\sum_{k\neq i}^K |(\bar{\mathbf{h}}_i + \epsilon_i \mathbf{v}_i)^H \boldsymbol{w}_k|^2 + \sigma_i^2} \ge \gamma_i \quad \forall \ \|\mathbf{v}_i\|^2 \le r_i^2,\\ & i = 1,\ldots,K. \end{aligned}$$

is a conservative formulation for problem (5). It is interesting to note from (12) that the probabilistic SINR constraints in (5) have been replaced by an infinite number of deterministic SINR constraints with spherically bounded uncertainty vectors $\{\mathbf{v}_i\}_{i=1}^K$. Problem (12) is reminiscent of the worst-case robust transmit beamforming problem considered in [2,3,7]. Different from [2,3,7] where the radii $\{r_i\}_{i=1}^K$ are preassigned system parameters, problem (12) now serves as a conservative formulation of the probabilistically-constrained problem in (5), and the radii $\{r_i\}_{i=1}^K$ are determined according to the SINR outage probabilities [see (11)].

3.2 SDR for Problem (12)

In this subsection, we present an SDR based method[1] for handling problem (12) efficiently.

To illustrate this method, let us define $\mathbf{W}_i \triangleq \boldsymbol{w}_i \boldsymbol{w}_i^H$ for $i = 1,\ldots,K$, and express (12) in terms of $\{\mathbf{W}_i\}_{i=1}^K$ as

$$\begin{aligned} \min_{\substack{\mathbf{W}_i \in \mathbb{C}^{N_t\times N_t},\\ i=1,\ldots,K}} \ & \sum_{i=1}^K \text{Tr}(\mathbf{W}_i) \qquad (13a)\\ \text{s.t.} \ & (\bar{\mathbf{h}}_i + \epsilon_i \mathbf{v}_i)^H \left(\frac{1}{\gamma_i}\mathbf{W}_i - \sum_{k\neq i}^K \mathbf{W}_k \right) (\bar{\mathbf{h}}_i + \epsilon_i \mathbf{v}_i)\\ & \ge \sigma_i^2, \ \forall \ \|\mathbf{v}_i\|^2 \le r_i^2, \ i = 1,\ldots,K, \qquad (13b)\\ & \mathbf{W}_i \succeq \mathbf{0}, \ \text{rank}(\mathbf{W}_i) = 1, \quad i = 1,\ldots,K, \qquad (13c) \end{aligned}$$

where Tr$(\cdot)$ denotes the trace of a matrix, $\mathbf{W}_i \succeq \mathbf{0}$ means that $\mathbf{W}_i$ is a Hermitian, positive semidefinite matrix, and the constraints in (13c) are due to the fact that $\mathbf{W}_i = \boldsymbol{w}_i \boldsymbol{w}_i^H$. The semi-infinite constraints in (13b) can be recast as a finite number of linear matrix inequalities, by applying the S-procedure [10]:

Lemma 2 (S-procedure [10]) Let $\mathbf{A} \in \mathbb{C}^{N_t\times N_t}$ be a complex Hermitian matrix, $\boldsymbol{b} \in \mathbb{C}^{N_t}$ and $c \in \mathbb{R}$. The following condition

$$\mathbf{v}^H \mathbf{A} \mathbf{v} + \boldsymbol{b}^H \mathbf{v} + \mathbf{v}^H \boldsymbol{b} + c \ge 0, \quad \forall \ \|\mathbf{v}\|^2 \le r^2$$

holds true if and only if there exists a nonnegative value $\lambda \ge 0$ such that

$$\begin{bmatrix} \mathbf{A} + \lambda \mathbf{I}_{N_t} & \boldsymbol{b} \\ \boldsymbol{b}^H & c - \lambda r^2 \end{bmatrix} \succeq \mathbf{0},$$

where $\mathbf{I}_{N_t}$ is the $N_t \times N_t$ identity matrix.

By applying Lemma 2 to the constraints in (13b), one can reformulate problem (13) as

$$\begin{aligned} \min_{\substack{\mathbf{W}_i \in \mathbb{C}^{N_t\times N_t}, \lambda_i \in \mathbb{R},\\ i=1\ldots,K}} \ & \sum_{i=1}^K \text{Tr}(\mathbf{W}_i) \qquad (14a)\\ \text{s.t.} \ & \boldsymbol{\Psi}_i(\mathbf{W}_1,\ldots,\mathbf{W}_K,\lambda_i) \succeq \mathbf{0},\\ & \mathbf{W}_i \succeq \mathbf{0}, \ \ \lambda_i \ge 0,\\ & \text{rank}(\mathbf{W}_i) = 1, \ \ i = 1,\ldots,K, \qquad (14b) \end{aligned}$$

where $\boldsymbol{\Psi}_i(\mathbf{W}_1,\ldots,\mathbf{W}_K,\lambda_i)$ is a linear matrix function defined as

$$\begin{aligned} \boldsymbol{\Psi}_i(\mathbf{W}_1,\ldots,\mathbf{W}_K,\lambda_i) \triangleq & \begin{bmatrix} \epsilon_i \mathbf{I}_{N_t} \\ \bar{\mathbf{h}}_i^H \end{bmatrix} \left(\frac{1}{\gamma_i}\mathbf{W}_i - \sum_{k\neq i}^K \mathbf{W}_k \right) \begin{bmatrix} \epsilon_i \mathbf{I}_{N_t} \\ \bar{\mathbf{h}}_i^H \end{bmatrix}^H \\ & + \begin{bmatrix} \lambda_i \mathbf{I}_{N_t} & \mathbf{0} \\ \mathbf{0} & -\sigma_i^2 - \lambda_i r_i^2 \end{bmatrix} \end{aligned} \tag{15}$$

for $i = 1,\ldots,K$. One can see that problem (14) has a linear objective function and convex constraints except for the nonconvex rank-one constraints in (14b). By removing these nonconvex constraints, we obtain the following problem

$$\begin{aligned} \min_{\substack{\mathbf{W}_i \in \mathbb{C}^{N_t\times N_t}, \lambda_i \in \mathbb{R},\\ i=1\ldots,K}} \ & \sum_{i=1}^K \text{Tr}(\mathbf{W}_i) \qquad (16)\\ \text{s.t.} \ & \boldsymbol{\Psi}_i(\mathbf{W}_1,\ldots,\mathbf{W}_K,\lambda_i) \succeq \mathbf{0},\\ & \mathbf{W}_i \succeq \mathbf{0}, \ \ \lambda_i \ge 0, \ i = 1,\ldots,K, \end{aligned}$$

as a relaxation to problem (12). The SDR problem (16) is a convex SDP, and thus can be efficiently solved by `CVX` [9].

The SDR technique is in general an approximation method in the sense that the optimum $\{\mathbf{W}_i\}_{i=1}^K$ of problem (16) may not have rank one. In that case, additional solution approximation procedures to turn the optimum $\{\mathbf{W}_i\}_{i=1}^K$ into a rank-one approximate solution of problem (12) is needed [11]. Interestingly, our experience in simulations indicates that the SDR problem (16) always yields rank-one solutions, which means that SDR provides globally optimum solutions of problem (12) for all the problem instances we tried in simulations. Similar empirical observations are also reported in [7] though a different robust beamforming problem is considered there. As a future direction, we will investigate the reason behind this intriguing finding. Here we are more interested in using this practically appealing result to help us resolve the conservative formulation (12).

3.3 Reducing the Level of Conservatism by Bisection

In the previous two subsections, we have shown that problem (12) with the radii $\{r_i\}_{i=1}^K$ given by (11) is a conservative formulation of the probabilistically-constrained problem (5), and we have discussed that problem (12) may be efficiently handled by SDR. It is noticed that problem (12) may be too conservative in the sense that the associated optimum beamfomers $\{\boldsymbol{w}_i\}_{i=1}^K$ may yield an SINR satisfaction probability much higher than $1-\rho_i$. Since the level of conservatism of problem (12) can be reduced by decreasing r_i, this motivates

[1]We should mention that the SDR based method presented in this subsection is inspired by [7] where the same technique is used to handle another robust transmit beamforming problem in cognitive radios.

us to iteratively update r_i in a bisection manner such that the level of conservatism can be reduced.

To illustrate this method, let us assume that all the receivers have the same SINR outage probability, i.e., $\rho \triangleq \rho_1 = \cdots = \rho_K$ and thus we can let $r \triangleq r_1 = \cdots = r_K$. In each iteration, we first solve problem (12) with a given r to obtain the associated optimum beamformers, denoted by $\{\boldsymbol{w}_i^\star\}_{i=1}^K$. Given $\{\boldsymbol{w}_i^\star\}_{i=1}^K$, one can apply the statistical validation procedure in [12] to determine whether the probabilistic SINR constraints in (5b) are (empirically) met or not. If yes, it means that there may exist room for further reducing r; otherwise one should increase r. Hence depending on the outcome of the validation procedure, the radius r can be updated in a bisection manner over the interval of $(0, \sqrt{\mathrm{ICDF}(1-\rho)/2}]$. In Table 1, we summarize the bisection procedure.

It should be mentioned that the above bisection procedure also applies to the conservative approaches presented in [4] where a different parameter that controls the level of conservatism can be iteratively updated via bisection; readers are referred to [4] for the details. As will be shown in the next section, the proposed new conservative approach (with bisection) can exhibit better performance than the methods in [4].

Table 1. Bisection procedure for the proposed conservative formulation (12).

Given	a desired SINR outage probability $\rho \in (0,1]$, and a solution accuracy $\varepsilon > 0$.
Set	$r_{\min} = 0$ and $r_{\max} = \sqrt{\mathrm{ICDF}(1-\rho)/2}$.
Step 1.	Solve problem (12) with $r = (r_{\max} + r_{\min})/2$ using the SDR technique in Section 3.2.
Step 2.	If (12) is feasible, let $\{\boldsymbol{w}_i^\star\}_{i=1}^K$ be the associated optimum solution, and go to Step 3; otherwise set $r_{\max} = r$ and go to Step 1.
Step 3.	Apply the statistical validation procedure in [12] to determine whether $\{\boldsymbol{w}_i^\star\}_{i=1}^K$ satisfies the probabilistic SINR constraints in (5b). Let $r_{\max} = r$ if (5b) is satisfied; otherwise set $r_{\min} = r$.
Step 4.	If $r_{\max} - r_{\min} \le \varepsilon$, then output $\{\boldsymbol{w}_i^\star\}_{i=1}^K$ as the desired beamforming vectors; otherwise go to Step 1.

4. SIMULATION RESULTS AND DISCUSSIONS

In this section, we present some simulation results to demonstrate the effectiveness of the proposed conservative approach. In the simulations, we consider a wireless system as described in Sec. 2 with the transmitter equipped with 3 antennas ($N_t = 3$) and 3 receivers ($K = 3$). The channel estimates $\{\bar{\mathbf{h}}_i\}_{i=1}^3$ are randomly generated according to i.i.d. complex Gaussian with zero mean and unit variance (i.e., $\bar{\mathbf{h}}_i \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_3)$). As mentioned in Sec. 2.2, the error vectors $\{\mathbf{e}_i\}_{i=1}^3$ are modeled as i.i.d. zero-mean complex Gaussian. For simplicity, all the entries of $\mathbf{e}_i$ have identical variance of 0.002 (i.e., $\epsilon_1^2 = \epsilon_2^2 = \epsilon_3^2 = 0.002$). The noise powers at receivers are also set the same: $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 0.01$. We assume that all the 3 receivers demand the same QoS requirement. In particular, the receivers' target SINR values $\{\gamma_i\}_{i=1}^3$ are set the same, i.e., $\gamma \triangleq \gamma_1 = \gamma_2 = \gamma_3$, and the SINR outage probabilities $\{\rho_i\}_{i=1}^3$ are all set as 0.1, i.e., $\rho \triangleq \rho_1 = \rho_2 = \rho_3 = 0.1$, which means that a 90% SINR satisfaction probability is desired. We applied the SDR technique in Sec. 3.2 to handle the proposed conservative formulation in (12) using the sphere radius r given in (11), which is labeled as 'Proposed method'. The SDP solver `CVX` [9] was used to solve problem (16). The bisection procedure in Table 1 was also tested (labeled as 'Proposed method with bisection') with the solution accuracy ε set as 10^{-3} ($\varepsilon = 10^{-3}$). To implement the statistical validation procedure in [12], we generated 92,103 sets of error vectors $\{\mathbf{e}_i\}_{i=1}^3$ for each realization of $\{\bar{\mathbf{h}}_i\}_{i=1}^3$, to test whether the probabilistic SINR constraints in (5b) are satisfied empirically. According to [12], this validation procedure has 99% confidence that the obtained empirical probability is correct. We compare the proposed conservative approach with the conservative formulation I proposed in [4] (labeled as 'Formulation I in [4]'). The method in [4] also admits a bisection procedure to reduce its level of conservatism. This procedure was also tested which is labeled as 'Formulation I in [4] with bisection'.

In the first example, we compare the feasibility rates of the proposed method and Formulation I in [4]. We generate 500 realizations of $\{\bar{\mathbf{h}}_i\}_{i=1}^3$, and, for each realization, we determine if the formulation under test is able to yield a solution $\{\boldsymbol{w}_i\}_{i=1}^3$ that has a finite power and satisfies the probabilistic SINR constraints in (5b) (which is verified by the statistical validation procedure in [12]). Figure 1 displays the simulation results of the feasibility rate (%) versus the target SINR γ (dB). As observed from this figure, the proposed method exhibits a much higher feasibility rate than Formulation I in [4] under the same γ. For example, for $\gamma = 17$ dB, the proposed method has around 30% feasibility rate; whereas Formulation I in [4] is hardly feasible. One can also see from this figure that the feasibility rates of both methods can be considerably increased when their respective bisection techniques are applied; however the proposed method (with bisection) still shows better performance.

To look further into the levels of conservatism of the proposed method and Formulation I in [4], we display in Fig. 2 the distribution (%) of the empirical SINR satisfaction probability of receiver 1 ($i = 1$). Note that only the realizations of $\{\bar{\mathbf{h}}_i\}_{i=1}^3$ for which the proposed method and Formulation I in [4] are both feasible are taken into account. Figures 2(a) and 2(b) show the results of the two methods (without bisection) for $\gamma = 13$ dB, respectively. One can see from the two figures that both methods are over conservative since the empirical satisfaction probabilities are much higher than the required probability 0.9. However, we can observe that the proposed method is slightly less conservative than Formulation I in [4]. Figures 2(c) to 2(f) display the results of the two methods with the bisection techniques applied. One can see that the levels of conservatism of both methods with bisection can be significantly reduced, while the proposed method with bisection is less conservative in a worst-case sense. For example, the maximum achieved SINR satisfaction probability in Fig. 2(e) is less than 0.965, whereas it is more than 0.985 in Fig. 2(f).

In the last example, we examine the required average transmission powers of the proposed method and the methods in [4]. To this end, we generate 500 realizations of $\{\bar{\mathbf{h}}_i\}_{i=1}^3$, and select the realizations for which the proposed method, Formulation I in [4] and their bisection counterparts are all able to satisfy the probabilistic SINR constraints in (5b) for $\gamma = 9$ dB. We obtain 201 such realizations of $\{\bar{\mathbf{h}}_i\}_{i=1}^3$ and use them to test the four methods for various values of target SINR γ. The average transmission power of each of the four methods is calculated over these 201 realizations, and are plotted in Fig. 3. Note that we set the average transmission power as infinity if there exist at least one realization of $\{\bar{\mathbf{h}}_i\}_{i=1}^3$ such that the method under test is infeasible. It can be seen from the figure that both the proposed method and the proposed method with bisection are more power efficient than the methods proposed in [4], and are able to support a wider range of target SINR values.

In summary, we have presented a new conservative ap-

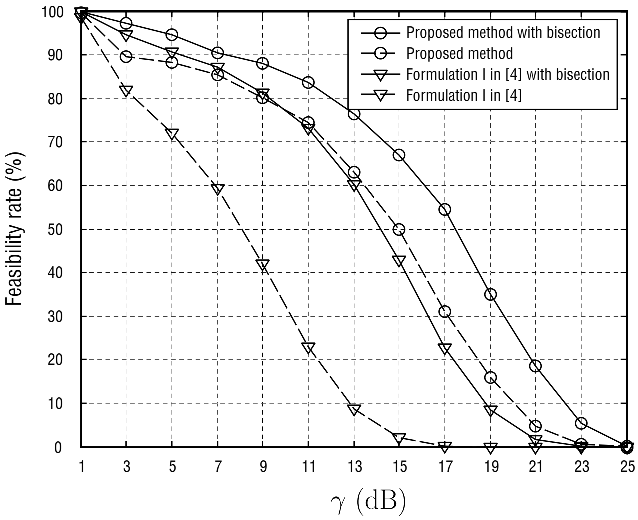

Figure 1. Simulation results of the feasibility rate (%) of the proposed method and Formulation I in [4].

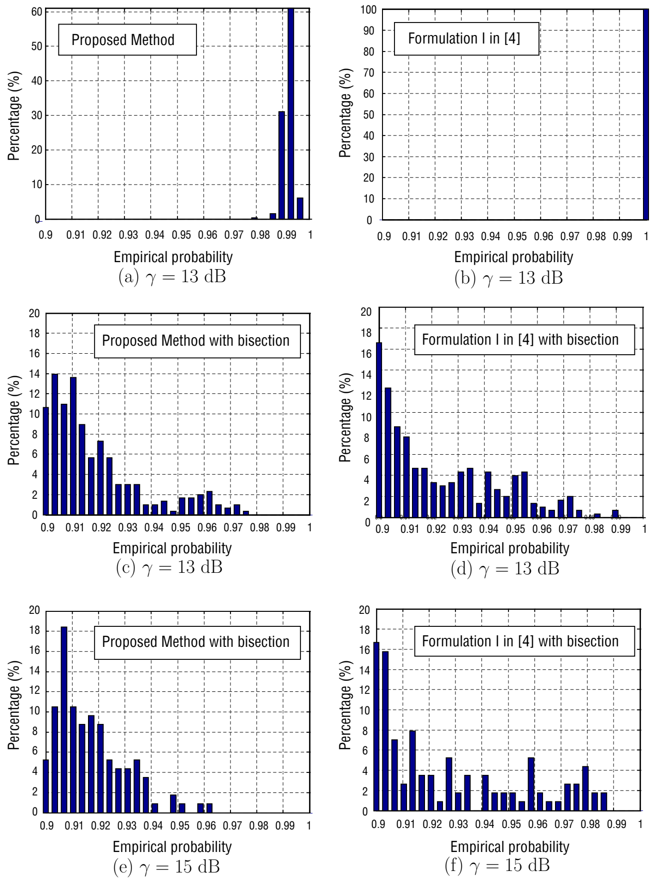

Figure 2. Simulation results of the distribution of empirical SINR satisfaction probability of receiver 1 obtained by the the proposed method and Formulation I in [4].

proach for the probabilistically-constrained robust transmit beamforming problem in (5). The proposed approach is based on the worst-case constrained conservative formulation in (12) which can be efficiently handled by SDR. The presented simulation results have shown that the proposed approach is less conservative and outperforms the existing methods.

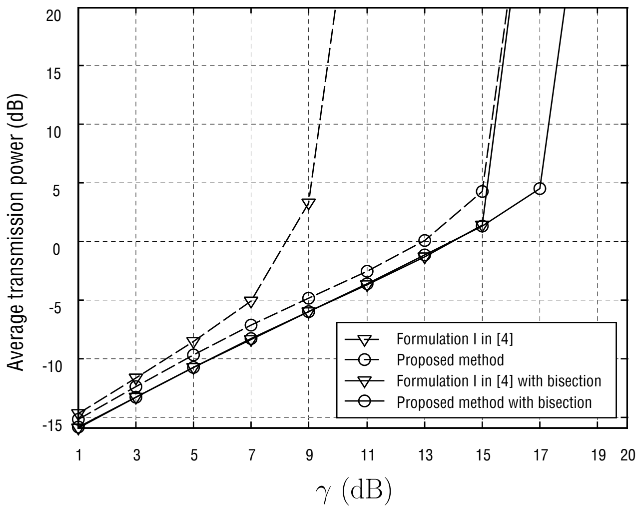

Figure 3. Simulation results of average transmission power of the proposed method and Formulation I in [4].

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