Peak Filter and Notch Filter for Harmonic Retrieval Using Higher-Order Statistics

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Abstract

In this paper, an algorithm using the well-known notch filter and an algorithm using a peak filter are proposed to estimate the frequencies of sinusoidal signals with a given set of Gaussian noise corrupted measurements $y(n)$ provided that the number of sinusoids is known in advance. The former processes $y(n)$ such that a single fourth-order cumulant of the notch filter output is minimum in absolute value, while the latter processes $y(n)$ such that the same fourth-order cumulant of the peak filter output is maximum in absolute value. Then the unknown frequencies are obtained from the optimum notch filter and the optimum peak filter, respectively. A performance analysis of the proposed two algorithms is then presented followed by some simulation results for a performance comparison of the proposed algorithms and Swami and Mendel's SVD low-rank approximation method.

1. Introduction

Estimation of parameters of sinusoidal signals is a problem to estimate frequencies $0 < \omega_i < \pi$ and amplitudes $A_i > 0$ with a given set of noisy measurements modeled as follows:

$$y(n) = \sum_{i=1}^{p} A_i cos(\omega_i n + \phi_i) + w(n) \qquad (1)$$

where p is the total number of sinusoids, ϕ_i's are random phases and $w(n)$ is additive noise. This is a well defined problem in some statistical signal processing areas such as noise and interference cancellation and estimation of direction of arrival (DOA) of narrowband source signals in sonar and radar arrays. Usually, frequency estimation is followed by amplitude estimation because the former often resorts to a nonlinear search procedure while the latter can be solved from a set of linear equations once ω_i's are estimated. There have been a number of correlation (second-order statistics) based algorithms reported for the estimation of ω_i's such as Pisarenko's harmonic decomposition procedure [1], Tufts and Kumaresan's method [2], overdetermined Yule-Walker method [3] and maximum-likelihood method [4]. Chicharo and Ng [5] proposed an adaptive notch filtering approach for the enhancement and tracking of sinusoids in additive noise. The transfer function of notch filters (IIR filters) of order equal to $2p$ is given by

$$H_p(z) = \frac{\prod_{i=1}^{p}(1 + \beta a_i z^{-1} + \beta^2 z^{-2})}{\prod_{i=1}^{p}(1 + \alpha a_i z^{-1} + \alpha^2 z^{-2})} \qquad (2)$$

where $0 \leq \beta \leq 1$ and $0 \leq \alpha < \beta$. The ω_i's are obtained by solving roots of the numerator polynomial of the adaptive notch filter.

Higher-order (≥ 3) statistics, known as cumulants, have been used for frequency estimation of sinusoidal signals when measurement noise is Gaussian because all higher-order cumulants of Gaussian noise are equal to zero. Thus cumulant based frequency estimation algorithms [6-8] are insensitive to additive Gaussian noise. In this paper, the notch filter and a peak filter, using a single fourth-order cumulant are proposed for frequency estimation of sinusoidal signals. A performance analysis of the proposed frequency estimation algorithms (one using the notch filter and the other using a peak filter) is presented followed by some simulation results.

2. Cumulant based harmonic retrieval using notch filters and peak filters

Assume that we are given a set of noisy measurements $y(n), n = 0, 1, \cdots, N-1$ modeled by (1) under the following assumptions:

(A1) The number p of sinusoids is known *a priori*; amplitudes $A_i > 0$ and frequencies $0 < \omega_i < \pi, i = 1, \cdots, p$ are unknown.

(A2) Measurement noise $w(n)$ is Gaussian with unknown statistics.

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(A3) Phase ϕ_i's are i.i.d. random variables with a uniform probability density function over $[-\pi, \pi)$ and they are statistically independent of $w(n)$.

Let $C_{M,e}(k_1, \cdots, k_{M-1})$ denote the Mth-order cumulant function of a non-Gaussian signal $e(n)$. We need the following proposition on which the two frequency estimation algorithms to be presented are based.

Proposition 1. Let $e(n)$ be the output of a linear time-invariant system $H(z)$ with input $y(n)$ given by (1) under the assumptions **(A1)** through **(A3)**, i.e.,

$$e(n) = y(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)y(n-k) \quad (3)$$

where $h(n)$ is the impluse response of the system. Then

$$C_{4,e}(0,0,0) = -\frac{3}{8}\sum_{i=1}^{p} A_i^4 \cdot |H(e^{j\omega_i})|^4 \quad (4)$$

A. Notch filter based algorithm:

By **Proposition 1**, one can infer the following fact:

(F1) Let $e(n)$ be the output signal given by (3) of the notch filter $H_p(z)$ with $\beta = 1$ given by (2). Then $|C_{4,e}(0,0,0)| = min\{|C_{4,e}(0,0,0)|\} = 0$ occurs only when $|H_p(e^{j\omega_i})| = 0$ for all i, i.e.,

$$a_i = -2 \cdot cos(\omega_i) \quad (5)$$

Let $\widehat{C}_{4,e}(0,0,0)$ denote the fourth-order sample cumulant associated with $C_{4,e}(0,0,0)$. By **(F1)**, we propose the following frequency estimation algorithm:

<u>**Algorithm 1**</u>:

(S1) Let $e(n)$ be the output signal given by (3) of the notch filter $H_p(z)$ ($\beta = 1$) given by (2). Find the optimum parameters $\widehat{a}_i, i = 1, \cdots, p$ of $H_p(z)$ such that $|\widehat{C}_{4,e}(0,0,0)|$ is minimum.

(S2) Obtain $\widehat{\omega}_i$ by (5), i.e.,

$$\widehat{\omega}_i = arccos(-\widehat{a}_i/2) \quad (6)$$

B. Peak filter based algorithm:

The peak filter used for frequency estimation is an IIR filter with transfer function

$$V_p(z) = \frac{\prod_{i=1}^{p}(1 + \rho\alpha a_i z^{-1} + \rho^2\alpha^2 z^{-2})}{\prod_{i=1}^{p}(1 + \alpha a_i z^{-1} + \alpha^2 z^{-2})} \quad (7)$$

where $0 < \alpha < 1$ and $0 \leq \rho < 1$. The peak filter differs from the notch filter in that each pair of complex conjugate poles (with magnitude α) are closer to the unit circle than the associated pair of complex conjugate zeros (with magnitude $\alpha\rho < \alpha$).

Again, by **Proposition 1**, one can infer the following fact:

(F2) Let $e(n)$ be the output signal given by (3) of the peak filter $V_p(z)$ given by (7). Then $|C_{4,e}(0,0,0)| = max\{|C_{4,e}(0,0,0)|\}$ occurs when $a_i, i = 1, \cdots, p$ of $V_p(z)$ are given by (5).

The following frequency estimation algorithm is due to **(F2)**:

<u>**Algorithm 2**</u>:

(S1) Let $e(n)$ be the output signal given by (3) of the peak filter $V_p(z)$ given by (7). Find the optimum parameters $\widehat{a}_i, i = 1, \cdots, p$ of $V_p(z)$ such that $|\widehat{C}_{4,e}(0,0,0)|$ is maximum.

(S2) Obtain $\widehat{\omega}_i$ using (6).

To find the optimum $\widehat{a}_i$ required in (S1) of the proposed two algorithms, we have to resort to iterative optimization algorithms because

$$\widehat{C}_{4,e}(0,0,0) = \frac{1}{N}\sum_{n=0}^{N-1} e^4(n) - 3\left(\frac{1}{N}\sum_{n=0}^{N-1} e^2(n)\right)^2 \quad (8)$$

is a highly nonlinear function of a_i. A gradient type iterative algorithm is used to search for the optimum $\mathbf{a} = (a_1, \cdots, a_p)^T$. At the nth iteration, $\widehat{\mathbf{a}}$ is updated by

$$\widehat{\mathbf{a}}(n) = \widehat{\mathbf{a}}(n-1) \pm \eta \frac{\partial|\widehat{C}_{4,e}(0,0,0)|}{\partial \mathbf{a}}\Big|_{\mathbf{a}=\widehat{\mathbf{a}}(n-1)} \quad (9)$$

where η is a small positive constant and "$-$" is for **Algorithm 1** and "$+$" is for **Algorithm 2**, respectively. An initial condition for $\widehat{\mathbf{a}}(0)$ is needed to initialize the iterative algorithm given by (9). Swami and Mendel's method [6] can be used to obtain an estimate for each ω_i and the associated a_i computed by (5) can be used for $\widehat{\mathbf{a}}(0)$.

3. Performance analysis

To illustrate the performance of the proposed two frequency estimation algorithms, let us assume that $p = 1$, $A_1 = 1$, $\omega_1 = 0.5\pi$ and $w(n)$ is white with variance σ_w^2. Then

$$|C_{4,e}(0,0,0)| = \begin{cases} (3/8)|H_1(e^{j\omega_1})|^4 & \text{for } \textbf{Algorithm 1} \\ (3/8)|V_1(e^{j\omega_1})|^4 & \text{for } \textbf{Algorithm 2} \end{cases}$$

with the same optimum solution $a_1 = a = 0$ by (5). Figure 1 (a) shows $log_{10}|C_{4,e}(0,0,0)|$ associated with the peak filter used by **Algorithm 2** for $\rho = 0.9$ and $\alpha = 0.9$ (dashed line), 0.95 (dotted line) and 0.99 (solid line), respectively, and Figure 1 (b) shows $|C_{4,e}(0,0,0)|$ instead of $log_{10}|C_{4,e}(0,0,0)|$ associated with the notch filter used by **Algorithm 1** for $\beta = 1$ and $\alpha = 0.9$ (dashed line), 0.95 (dotted line) and 0.99 (solid line), respectively. One can see, from these two figures, that a single peak (whose magnitude is larger for larger α) in Figure 1 (a) and a single notch ($|C_{4,e}(0,0,0)| = 0$) in Figure 1 (b) located at $a = 0$ are associated with each curve, and that the larger α, the narrower is the peak for the former and the notch for the latter.

It can be shown that

$$\widehat{C}_{4,e}(0,0,0) \approx C_{4,e}(0,0,0) + \widehat{C}_{4,w'}(0,0,0) \qquad (10)$$

where $\widehat{C}_{4,w'}(0,0,0)$ is the fourth-order sample cumulant of the Gaussian noise $w'(n)$ in the filter output $e(n)$ due to the presence of $w(n)$. Note that $\widehat{C}_{4,w'}(0,0,0)$ itself is a random variable. For the notch filter, it can be shown that for $a = 0$

$$E[\widehat{C}^2_{4,w'}(0,0,0)] \geq \sigma_1^2 = \left[\frac{6}{N}\sum_{l=-N}^{N}(1-\frac{|l|}{N})r^2_{w'}(l)\right]^2$$

where $r_{w'}(l)$ (autocorrelation function of $w'(n)$) is given by

$$r_{w'}(l) = \begin{cases} \sigma_w^2 \cdot \frac{2}{1+\alpha^2}, & l = 0 \\ \sigma_w^2 \cdot \frac{(\alpha^2-1)\alpha^{|l|}cos(l\pi/2)}{(1+\alpha^2)\alpha^2}, & l \neq 0 \end{cases}$$

Therefore, $min\{|C_{4,e}(0,0,0)|\} = 0$ is easily smeared by $\widehat{C}_{4,w'}(0,0,0)$ if $\sigma_1 \gg 0$ (low SNR). On the other hand, for the peak filter, it can be shown that for $a = 0$

$$E[\widehat{C}^2_{4,w'}(0,0,0)] \leq \sigma_2^2 = 1050 \cdot \left(1 + \frac{(\rho^2-1)^2\alpha^4}{1-\alpha^4}\right)^4 \cdot \sigma_w^8$$

One can easily infer that if $max\{|C_{4,e}(0,0,0)|\}/\sigma_2 = (3/8)|V_1(e^{j0.5\pi})|^4/\sigma_2 \gg 1$, the optimum $a = 0$ can be accurately estimated even if SNR is low. For instance, $max\{|C_{4,e}(0,0,0)|\} = 4316 \gg \sigma_2 = 28.6$ for $SNR = 0\ dB$, $\rho = 0.9$ and $\alpha = 0.99$. Therefore, the previous performance analysis leads to following fact:

(F3) **Algorithm 2** outperforms **Algorithm 1** for finite data, because the former is more robust to additive noise than the latter.

4. Simulation results

As mentioned in Section 2, SM method [6] was used to provide an initial condition for the proposed two frequency estimation algorithms. In the simulation, thirty independent runs were performed to compute the mean square error (MSE) defined as

$$MSE = \frac{1}{30}\sum_{j=1}^{30}\{\sum_{i=1}^{p}(\widehat{f}_{ij} - f_i)^2\} \qquad (11)$$

where $f_i = \omega_i/2\pi$ and $\widehat{f}_{ij}$ is the obtained estimate for f_i at the jth run. Two sets of simulation results ($p = 1$ and $p = 2$, $A_1 = A_2$) for measurement noise $w(n)$ assumed to be white Gaussian were obtained using **Algorithm 1** with $\beta = 1$ and $\alpha = 0.99$ and **Algorithm 2** with $\rho = 0.9$ and $\alpha = 0.99$, respectively.

Let $SNR = A_i^2/(2\sigma_w^2)$ where σ_w^2 is the variance of $w(n)$. Table 1 shows the simulation results for $p = 1$, $A_1 = 1$, $f_1 = 0.2$, $N = 1024$, 2048, 4096 and $SNR = 0$, 5, 10, 15, 20 dB. From this table, one can see that **Algorithm 2** performs best, SM method performs second and **Algorithm 1** performs worst. On the other hand, Table 2 shows the corresponding results for $p = 2$, $A_1 = A_2 = 1$, $f_1 = 0.1$ and $f_2 = 0.2$. From Table 2, one can see that **Algorithm 2** performs best except for the case that $SNR = 0\ dB$ when $N = 1024$ and 2048 while SM method performs best for this case. These simulation results indicate that the latter may perform better than the former for small N and low SNR. However, **Algorithm 1** always performs worst as predicted by **(F3)**, and its performance for low SNR may not improve even when N is increased (see the results for $N = 2048$ and 4096 when $SNR = 0\ dB$, 5 dB and 10 dB in Table 2). The reason for this is that although N was doubled, the notch of $min\{|C_{4,e}(0,0,0)|\} = 0$ in some realizations was severely smeared by $\widehat{C}_{4,w'}(0,0,0) \approx \widehat{C}_{4,e}(0,0,0)$ at the vicinity of $(a_1, a_2)^T = (-2cos(0.2\pi),\ -2cos(0.4\pi))^T$ where $w'(n)$ was the Gaussian noise in the notch filter output due to measurement noise $w(n)$.

5. Conclusions

We have presented two frequency estimation algorithms with a given set of noisy sinusoidal signals under the three assumptions **(A1)** through **(A3)**. **Algorithm 1** uses the notch filter and **Algorithm 2** uses the peak filter, while the former tries to minimize but the latter tries to maximize the same single absolute fourth-order cumulant. A performance analysis for the proposed two algorithms was also presented. Then some simulation results obtained by the proposed two algorithms and Swami and Mendel's method were presented for a performance comparison. The presented simulation results support that **Algorithm 2** performs best for the case of $p = 1$, but for the case of $p = 2$ it performs best except that when N is small and SNR is low, Swami and Mendel's method performs best.

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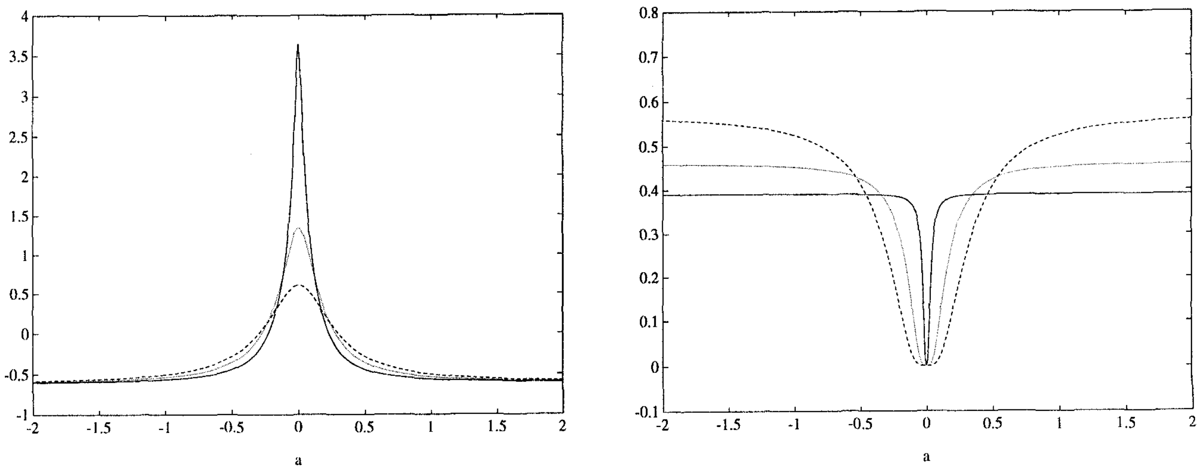

(a) (b)

Figure 1. (a) $log_{10}|C_{4,e}(0,0,0)|$ associated with the peak filter for $\rho = 0.9$ and $\alpha = 0.9$ (dashed line), 0.95 (dotted line) and 0.99 (solid line), respectively; (b) $|C_{4,e}(0,0,0)|$ associated with the notch filter for $\beta = 1$ and $\alpha = 0.9$ (dashed line), 0.95 (dotted line) and 0.99 (solid line), respectively.

N	SNR	$MSE(\times 10^{-7})$		
		SM Method	**Algori -thm 2**	**Algori -thm 1**
1024	20 dB	0.0171	0.0026	0.0471
	15 dB	0.0266	0.0122	0.1401
	10 dB	0.0634	0.0310	0.3955
	5 dB	0.3141	0.1838	1.6149
	0 dB	4.4869	2.3268	8.3789
2048	20 dB	0.0026	0.0003	0.0478
	15 dB	0.0059	0.0026	0.1439
	10 dB	0.0214	0.0112	0.5197
	5 dB	0.1717	0.0856	1.7674
	0 dB	3.7787	1.6415	8.2338
4096	20 dB	0.0010	0.0003	0.0199
	15 dB	0.0019	0.0004	0.0617
	10 dB	0.0060	0.0015	0.1888
	5 dB	0.0480	0.0152	0.5698
	0 dB	0.9703	0.3434	1.7159

Table 1. MSE's associated with the SM method, **Algorithm 1** (using the notch filter) and **Algorithm 2** (using the peak filter) for $p = 1$ and $f_1 = 0.2$.

N	SNR	$MSE(\times 10^{-7})$		
		SM Method	**Algori -thm 2**	**Algori -thm 1**
1024	20 dB	1.4042	0.0035	0.5792
	15 dB	1.5070	0.0041	2.0163
	10 dB	1.8507	0.1120	100.93
	5 dB	4.8170	0.0266	1394.9
	0 dB	86.835	148.69	3034.0
2048	20 dB	0.2858	0.0010	0.5308
	15 dB	0.3233	0.0011	1.7446
	10 dB	0.5162	0.0023	42.250
	5 dB	2.3045	0.0066	269.14
	0 dB	39.381	41.777	1605.0
4096	20 dB	0.0974	0.0002	0.5556
	15 dB	0.1133	0.0005	1.8608
	10 dB	0.2016	0.0006	239.27
	5 dB	0.9268	0.0013	518.37
	0 dB	10.896	0.0035	10596

Table 2. MSE's associated with the SM method, **Algorithm 1** (using the notch filter) and **Algorithm 2** (using the peak filter) for $p = 2$, $f_1 = 0.1$ and $f_2 = 0.2$.