EE306001 Probability, Fall 2012

Quiz #9, Problems and Solutions

Prob. 1: A probability density function of X is given by

$$f(x) = \begin{cases} ax^2, & 0 \leq x \leq 1, \\ b, & 1 < x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

If $E[X] = 1$, find a and b.

Solution: Since $f(x)$ is a probability density function, we have

$$\begin{aligned} 1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 ax^2 dx + \int_1^2 bdx \\ = & \frac{1}{3}ax^3 \mid_0^1 + bx \mid_1^2 \\ = & \frac{1}{3}a + b. \end{aligned}$$

Since $E[X] = 1$, we have

$$\begin{aligned} 1 = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 xax^2 dx + \int_1^2 xbdx \\ = & \frac{1}{4}ax^4 \mid_0^1 + \frac{1}{2}bx^2 \mid_1^2 \\ = & \frac{1}{4}a + \frac{3}{2}b. \end{aligned}$$

Thus, we have

$$\begin{cases} \frac{1}{3}a + b = 1 \\ \frac{1}{4}a + \frac{3}{2}b = 1 \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = \frac{1}{3}. \end{cases}$$

Prob. 2: Let X be a random variable.

(a) Prove that $\text{Var}(aX + b) = a^2 \text{Var}(X)$.

(b) Define $\text{SD}(X) = \sqrt{\text{Var}(X)}$. Find $\text{SD}(aX + b)$ if $\text{SD}(X) = \sigma$.

Solution:

(a)

$$\begin{aligned} \text{Var}(aX + b) = & E[(aX + b - E[aX + b])^2] = E[(aX + b)^2] - (E[aX + b])^2 \\ = & E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2 \\ = & a^2E[X^2] + 2abE[X] + b^2 - a^2(E[X])^2 - 2abE[X] - b^2 \\ = & a^2(E[X^2] - (E[X]^2)) \\ = & a^2 \text{Var}(X). \end{aligned}$$

(b)

$$\begin{aligned}\mathrm{SD}(aX+b) =& \sqrt{\mathrm{Var}(aX+b)} = \sqrt{a^2\mathrm{Var}(X)} \\ =& |a|\sqrt{\mathrm{Var}(X)} = |a|\mathrm{SD}(X) = |a|\sigma.\end{aligned}$$

Prob. 3: Let X be an exponential random variable with parameter $\lambda > 0$ and pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Let $g(x) = ax^2 + bx + c$. Compute $E[g(X)]$.

Solution:

By the linear property of expectation,

$$E[g(X)] = E[aX^2 + bX + c] = aE[X^2] + bE[X] + c,$$

where

$$\begin{aligned}E[X] &= \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} \lambda x e^{-\lambda x} dx \quad \text{(Use integration by part)} \\ &= -xe^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx \\ &= -0 + 0 - \frac{1}{\lambda} e^{-\lambda x}\Big|_{x=0}^{x=\infty} = \frac{1}{\lambda}; \\ E[X^2] &= \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^{\infty} \lambda x^2 e^{-\lambda x} dx \quad \text{(Use integration by part)} \\ &= -x^2 e^{-\lambda x}\Big|_0^{\infty} + 2\int_0^{\infty} xe^{-\lambda x} dx \\ &= -0 + 0 + \frac{2}{\lambda^2}.\end{aligned}$$

Therefore

$$E[g(X)] = aE[X^2] + bE[X] + c = \frac{2a}{\lambda^2} + \frac{b}{\lambda} + c.$$