EE306001 Probability, Fall 2012

Quiz #8, Problems and Solutions

Notation: The integration we discuss here is Riemann-Stieltjes integral.

Prob. 1: (20%)

(1) Let $F(x) = 1 - e^{-x}$. Compute $\int_{[0,\infty)} x dF$.

(2) Let $u_0(x)$ be the unit step function; $F(x) = 2u_0(x-3) + 4u_0(x-6) + u_0(x-8)$. Compute $\int_{[3,8]} \sqrt{x} dF$.

Solution:

(1) Let $b > 0$. For the interval $[0, b]$, $f(x) = x$ is continuous on $[0, b]$. And F', the derivative of F, exists on $[0, b]$ with $F'(x) = e^{-x} > 0$ on $[0, b]$, thus F is monotone increasing on $[0, b]$. Therefore the integral $\int_{[0,b]} x dF(x)$ exists and we have

$$\int_{[0,b]} x dF(x) = \int_{[0,b]} x e^{-x} dx = -be^{-b} - e^{-b} + 1.$$

Since $\lim_{b \to \infty} \int_{[0,b]} x dF(x) = 1$ exists and is finite, we have

$$\int_{[0,\infty)} x dF(x) = \lim_{b \to \infty} \int_{[0,b]} x dF(x) = 1.$$

(2) Let $f(x) = \sqrt{x}$. Since F is monotone increasing and f is continuous on $[3, 8]$, the integral $\int_{[3,8]} f dF$ exists and

$$\int_{[3,8]} f dF = 4 \cdot f(6) + 1 \cdot f(8) = 4\sqrt{6} + 2\sqrt{2}.$$

Prob. 2: (10%) Let $F(x) = e^x$. Does the improper Riemann-Stieltjes integral $\int_{[0,\infty)} x dF(x)$ exist ? If exists, is it finite ? If finite, what is its value ?

Solution:

Since x is continuous on $[0, b]$ for each $b \geq 0$, the Riemann-Stieltjes integral $\int_0^b x dF(x)$ exists and is finite. Since x is nonnegative on $[0, \infty)$, we have

$$\int_0^b x dF(x) \leq \int_0^b x dF(x) + \int_b^{b'} x dF(x) = \int_0^{b'} x dF(x)$$

for all $0 \leq b < b'$ so that $\int_0^b x dF(x)$ is monotone increasing as b increases. Thus the limit

$$\lim_{b \to \infty} \int_0^b x dF(x)$$

exists, i.e., the improper Riemann-Stieltjes integral $\int_0^\infty x dF(x)$ exists. With $F(x) = e^x$, by the integration by part, we have

$$\int_0^b x dF(x) = xF(x)\Big|_{x=0}^{x=b} - \int_0^b F(x)dx = be^b - e^x\Big|_0^b = e^b(b-1) + 1.$$

It's obvious that the limit

$$\lim_{b\to\infty} \int_0^b x dF(x) = \lim_{b\to\infty} e^b(b-1) + 1 = \infty$$

is infinite so that the improper Riemann-Stieltjes integral $\int_0^\infty x dF(x)$ is not finite.