EE306001 Probability, Fall 2012

Quiz #7, Problems and Solutions

Prob. 1: Let X be a continuous random variable whose pdf is given by

$$f(x) = \begin{cases} Ce^{-\lambda x} & \text{, if } x \geq 0 \\ 0 & \text{, if } x < 0 \end{cases},$$

where the parameter λ is greater than 0 and C is a constant.

(1) (5%) What is the value of C?

(2) (5%) What is the cumulative distribution function $F(x) = \mathcal{P}(X \leq x)$?

(3) (10%) Show that $\mathcal{P}(X > s + t | X > s) = \mathcal{P}(X > t)$, $\forall s, t \geq 0$.

Solution:

(1) The pdf must satisfy

$$\begin{aligned} 1 = \mathcal{P}(\mathbb{R}) &= \int_{-\infty}^{\infty} f(x) dx \\ &= \int_{0}^{\infty} Ce^{-\lambda x} \\ &= -\frac{C}{\lambda} e^{-\lambda x} \Big|_{x=0}^{x=\infty} = \frac{C}{\lambda}. \end{aligned}$$

Thus $C = \lambda$.

(2) The CDF is

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(u) du \\ &= \int_{0}^{x} \lambda e^{-\lambda u} du \\ &= -e^{\lambda x} + 1 \end{aligned}$$

for $x \geq 0$ and $F(x) = 0$ for $x < 0$.

(3) For $s \geq 0$, we have

$$\mathcal{P}(X > s) = \int_{s}^{\infty} \lambda e^{-\lambda x} dx = e^{-\lambda s}.$$

Thus for $s, t \geq 0$, we have

$$\begin{aligned} \mathcal{P}(X > s + t | X > s) &= \frac{\mathcal{P}(X > s + t, X > s)}{\mathcal{P}(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= \mathcal{P}(X > t). \end{aligned}$$

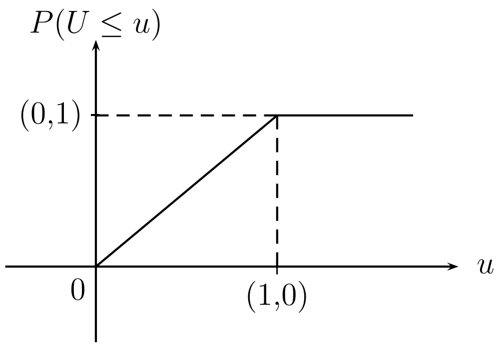
Prob. 2: Let U be a uniform random variable on $(0, 1)$.

(a) Please sketch the cumulative distribution function and a probability density function of U.

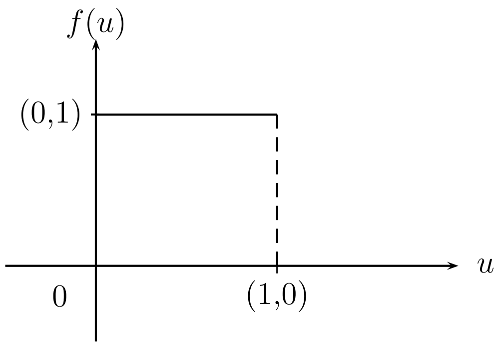
(b) For any nonnegative real number x, let $[x]$ be the largest integer that is less than or equal to x. Define the random variable $X = [nU] + 1$. Please find the probability $P(X = i)$, $1 \leq i \leq n$.

Solution:

(a) The cumulative distribution function is:



A probability density function is:



(b)

$$\begin{aligned}
P(X = i) =& P([nU] + 1 = i) = P([nU] = i - 1) \\
=& P(i - 1 \leq nU < i) = P(\frac{i-1}{n} \leq U < \frac{i}{n}) \\
=& \int_{(i-1)/n}^{i/n} f(u)du = u \mid_{(i-1)/n}^{i/n} \\
=& \frac{i}{n} - \frac{i-1}{n} = \frac{1}{n}.
\end{aligned}$$