# EE306001 Probability, Fall 2012

## Quiz #6, Problems and Solutions

**Prob. 1:** A fair coin is continually flipped until heads appears for the 10th time. Let $X$ denote the number of tails that occur.

(a) Construct a probability space $(S, \mathcal{F}, P)$ for this experiment. (3 pts)

(b) Show that $X$ is a random variable based on the probability space you constructed in (a). (3 pts)

(c) Compute the probability mass function of $X$. (4 pts)

**Solution:**

(a) Model the coin as a set

$$C = \{H, T\}.$$

The sample space of this experiment can be defined as

$$S = C^\infty = C \times C \times \cdots .$$

The $\sigma$-algebra $\mathcal{F}$ is set to be the smallest $\sigma$-algebra containing the following subsets of $S$

$$E_n(a) \triangleq \{(\omega_1, \omega_2, \ldots, \omega_n, \ldots) \in S \,|\, \omega_n = a, \omega_i \in C \ \forall\ i \neq n\}$$

for all $n \geq 1$ and all $a \in C$. To reflect that the trials of tossing the coin are independent and the coin is fair, the probability function $P : \mathcal{F} \to \mathbb{R}^+$ has the following assignment

$$P(E_1(a_1) \cap E_2(a_2) \cap \cdots \cap E_n(a_n)) = \frac{1}{2^n}$$

for all $n \geq 1$ and all $a_1, a_2, \ldots, a_n \in C$.

(b) First $X : S \to \mathbb{R}$ is a real-valued function on $S$ whose range is the set of all nonnegative integers. To show that $X$ is a discrete r.v., we have to show that $(X = i) = \{\omega \in S | X(\omega) = i\}$ is an event, i.e., $(X = i) \in \mathcal{F}$, for all integers $i \geq 0$. But for any $i \geq 0$, we have

$$\begin{aligned} & (X = i) \\ = \ & (\omega_1, \omega_2, \ldots, \omega_{i+9}, H, \ldots) \in S \,|\ \text{9 of } \omega_1, \ldots, \omega_{9+i} \text{ are } H\text{'s and } \omega_n \in C \ \forall\ n \geq i + 10\} \\ = \ & \biguplus_{\text{9 of } \omega_1, \ldots, \omega_{9+i} \text{ are } H\text{'s}} (E_1(\omega_1) \cap \cdots \cap E_{i+9}(\omega_{i+9}) \cap E_{i+10}(H)), \end{aligned}$$

which shows that $(X = i) \in \mathcal{F}$ and then $X$ is a discrete random variable.

(c) From (b), we have

$$\begin{aligned}
P(X=i) &= P(\uplus_{9 \text{ of } \omega_1,\ldots,\omega_{9+i} \text{ are } H\text{'s}}(E_1(\omega_1)\cap\cdots\cap E_{i+9}(\omega_{i+9})\cap E_{i+10}(H))) \\
&= \sum_{9 \text{ of } \omega_1,\ldots,\omega_{9+i} \text{ are } H\text{'s}} P(E_1(\omega_1)\cap\cdots\cap E_{i+9}(\omega_{i+9})\cap E_{i+10}(H)) \\
&= \sum_{9 \text{ of } \omega_1,\ldots,\omega_{9+i} \text{ are } H\text{'s}} \frac{1}{2^{10+i}} \\
&= \binom{i+10-1}{10-1}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{i},
\end{aligned}$$

$i=0,1,2,\ldots$. This shows that $X$ is a negative binomial r.v. with parameters $r=10$ and $p=1/2$.

**Prob. 2:** For a nonnegative integer-valued random variable $N$, show that

$$E[N]=\sum_{i=1}^{\infty}P(N\geq i).$$

**Solution:**

$$\begin{aligned}
\sum_{i=1}^{\infty}P(N\geq i) &= \sum_{i=1}^{\infty}\sum_{k=i}^{\infty}P(N=k) \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{k}P(N=k) \\
&= \sum_{k=1}^{\infty}kP(N=k) \\
&= E[N].
\end{aligned}$$

**Prob. 3:** On average, there are 3.5 typhoons which hit a certain region of Taiwan in a year. What is the probability that there will be 3 or more typhoons hitting next year? (Note that the number of typhoons which hit that region of Taiwan in a year can be modeled as a Poisson random variable. Also note that $e^{-3.5}\doteqdot 0.03$.)

**Solution:** Let $X$ be a Poisson random variable with parameter $\lambda=3.5$. Note that $E[X]=\lambda=3.5$. Hence

$$\begin{aligned}
\mathcal{P}(X\geq 3) &= 1-(\mathcal{P}(X=0)+\mathcal{P}(X=1)+\mathcal{P}(X=2)) \\
&= 1-e^{-\lambda}(1+\lambda+\frac{\lambda^2}{2}) \\
&\doteqdot 1-0.03(1+3.5+6.125)=0.68125.
\end{aligned}$$