EE306001, Probability, Fall 2012

Quiz #5, Problems and Solutions

Prob. 1: Suppose that n independent trials, each of which results in any of the outcomes 0, 1, or 2, with respective probabilities p_0, p_1, and p_2, $\sum_{i=0}^{2} p_i = 1$, are performed.

(a) Construct a probability space $(S, \mathcal{F}, P)$ for this experiment. (You have to use formal representation of set when representing the sample space S, and specify your probability function clearly.)(4 pt)

(b) According to your probability space, what is the probability that outcome 1 never occur?(3 pt)

(c) According to your probability space, what is the probability that outcomes 1 and 2 both occur at least once?(3 pt)

Solution:

(a) Let

$$A = \{0, 1, 2\}$$

be the set of all possible outcomes of a single trial. The sample space of n trials is defined as

$$S = \underbrace{A \times A \times \cdots \times A}_{n} = A^n.$$

The σ-algebra is set to be

$$\mathcal{F} = 2^S.$$

Let the event

$$E_{ij} = \underbrace{A \times \cdots \times A}_{i-1} \times \{j\} \times \underbrace{A \times \cdots \times A}_{n-i}, \ \forall i, j \in \mathbb{Z}, 1 \leq i \leq n, 0 \leq j \leq 2.$$

According to the question, the probability function has to have

$$P(E_{i,j}) = p_j, \ \forall i, j \in \mathbb{Z}, 1 \leq i \leq n, 0 \leq j \leq 2.$$

Furthermore, $E_{1,j_1}, E_{2,j_2}, \ldots, E_{n,j_n}$ are independent events for any $J_1, j_2, \ldots, j_n \in A$, i.e.,

$$P(E_{1,j_1} \cap E_{2,j_2} \cap \ldots \cap E_{n,j_n}) = P(E_{1,j_1})P(E_{2,j_2}) \cdots P(E_{n,j_n}).$$

Therefore for any $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in S$, we have

$$\begin{aligned} P(\{\omega\}) =& P(\{(\omega_1, \omega_2, \ldots, \omega_n)\}) \\ =& P(E_{1,\omega_1} \cap E_{2,\omega_2} \cap \cdots \cap E_{n,\omega_n}) \\ =& P(E_{1,\omega_1})P(E_{2,\omega_2}) \cdots P(E_{n,\omega_n}) \\ =& p_{\omega_1} p_{\omega_2} \cdots p_{\omega_3}. \end{aligned}$$

And, for any $E \in \mathcal{F}$,

$$P(E) = \sum_{\omega \in E} P(\{\omega\}).$$

(b) Let

$$E_b = \text{the event that outcome 1 never occur,}$$

Then,

$$\begin{aligned}P(E_b) =& P(\bigcap_{i=1}^{n}(E_{i0} \cup E_{E_{i2}})) \\ =& \prod_{i=1}^{n} P(E_{i0} \cup E_{i2}) \\ =& (p_0 + p_2)^n.\end{aligned}$$

(c) Let

$$\begin{aligned}E_c =& \text{the event that both outcomes 1 and 2 occur at least once,} \\ E_1 =& \text{the event that outcomes 1 occurs at least once,} \\ E_2 =& \text{the event that outcomes 2 occurs at least once.}\end{aligned}$$

Then

$$\begin{aligned}P(E_c) =& P(E_1 \cap E_2) \\ =& 1 - P(E_1^c \cup E_2^c) \\ =& 1 - (P(E_1^c) + P(E_2^c) - P(E_1^c \cap E_2^c)).\end{aligned}$$

By (b), we have

$$P(E_1^c) = (p_0 + p_2)^n.$$

Similarly,

$$P(E_2^c) = (p_0 + p_1)^n.$$

Besides,

$$P(E_1^c \cap E_2^c) = P(\bigcap_{i=1}^{n} E_{i3}) = p_3^n.$$

Therefore,

$$P(E_c) = 1 - ((p_0 + p_2)^n + (p_0 + p_1)^n - p_3^n).$$

Prob. 2:

If E_1 and E_2 are conditionally independent given F, and $\mathcal{P}(E_1 \cap F) > 0$. Show that

$$\mathcal{P}_F(E_2|E_1) = \mathcal{P}_{E_1 \cap F}(E_2) = \mathcal{P}_F(E_2).$$

Solution.

$$
\begin{aligned}
\mathcal{P}_F(E_2|E_1) &= \frac{\mathcal{P}_F(E_1 \cap E_2)}{\mathcal{P}_F(E_1)} \\
&= \Big(\frac{\mathcal{P}(E_1 \cap E_2 \cap F)}{\mathcal{P}(F)}\Big) \Big/ \Big(\frac{\mathcal{P}(E_1 \cap F)}{\mathcal{P}(F)}\Big) \\
&= \frac{\mathcal{P}\big(E_2 \cap (E_1 \cap F)\big)}{\mathcal{P}(E_1 \cap F)} \\
&= \mathcal{P}_{E_1 \cap F}(E_2). \\
\text{And } \ \mathcal{P}_F(E_2|E_1) &= \frac{\mathcal{P}_F(E_1 \cap E_2)}{\mathcal{P}_F(E_1)} \\
&= \frac{\mathcal{P}_F(E_1) \cdot \mathcal{P}_F(E_2)}{\mathcal{P}_F(E_1)} \\
&= \mathcal{P}_F(E_2).
\end{aligned}
$$

Prob. 3:

Prove or disprove with a counterexample:

(1) If E and F are independent, then E and F are conditionally independent given every event $G \in \mathcal{F}$.

(2) If there is an event $G \in \mathcal{F}$ such that E and F are conditionally independent given G, then E and F are independent.

Solution.

(1) If we throw a fair dice twice. Let $S = \{(x_1, x_2)|x_i = 1, 2, 3, 4, 5, 6\}$ be the sample space, $\mathcal{F} = 2^S$, and $\mathcal{P}(E) = \frac{|E|}{|S|}$. Let $E = \{x_1 = 1\}$, $F = \{x_2 = 1\}$, and $G = \{x_1 + x_2 = 5\}$. Then $\mathcal{P}(E \cap F) = \mathcal{P}(E) \cdot \mathcal{P}(F) = \frac{1}{36}$ implies that E and F are independent. But $\mathcal{P}_G(E) = \mathcal{P}_G(F) = \frac{1}{4}$, $\mathcal{P}_G(E \cap F) = 0$, E and F aren't conditionally independent given G.

(2) If we throw a fair dice twice. Let $S = \{(x_1, x_2)|x_i = 1, 2, 3, 4, 5, 6\}$ be the sample space, $\mathcal{F} = 2^S$, and $\mathcal{P}(E) = \frac{|E|}{|S|}$. Let $E = \{x_1 + x_2 = 2\}$, $F = \{x_2 = 1\}$, and $G = \{x_1 + x_2 = 5\}$. Then $\mathcal{P}_G(E \cap F) = \mathcal{P}_G\big((1, 1)\big) = 0 = \mathcal{P}_G(E) = \mathcal{P}_G(E) \cdot \mathcal{P}_G(F)$, E and F are conditionally independent given G. But $\mathcal{P}(E \cap F) = \frac{1}{36}$, $\mathcal{P}(E) = \frac{1}{36}$, $\mathcal{P}(F) = \frac{1}{6}$. $\mathcal{P}(E \cap F) \neq \mathcal{P}(E) \cdot \mathcal{P}(F)$ implies that E and F are not independent.

Notation.

(1) For two events E and F, denote $\mathcal{P}_F(E) \triangleq \mathcal{P}(E|F)$.

(2) For three events E_1, E_2, and F, we say E_1 and E_2 are conditionally independent given F if $\mathcal{P}_F(E_1 \cap E_2) = \mathcal{P}_F(E_1) \cdot \mathcal{P}_F(E_2)$.