EE306001, Probability, Fall 2012

Quiz #4, Problems and Solutions

Prob. 1: Barbara and Dianne go target shooting. Suppose that each of Barbara's shots hits a wooden duck target with probability p_1, while each shot of Dianne's hits it with probability p_2. Suppose that they shoot simultaneously at the same target. If the wooden duck is knocked over (indicating that it was hit), what is the probability that

(a) both shots hit the duck?

(b) Barbara's shot hit the duck?

(You have to specify the sample space S, the σ-algebra $\mathcal{F}$,the probability function P and the independent assumptions you made.)

Solution: Let the sample space

$$\begin{aligned} S = \{&(\text{Barbara hit}, \text{Dianne hit}) \\ &(\text{Barbara hit}, \text{Dianne miss}) \\ &(\text{Barbara miss}, \text{Dianne hit}) \\ &(\text{Barbara miss}, \text{Dianne miss})\}, \end{aligned}$$

and the σ-algebra

$$\mathcal{F} = 2^S$$

let $P : \mathcal{F} \to \mathbb{R}$ be the probability function. According to the question, we have

$$\begin{aligned} &P(\{\text{Barbara hit}\}) \\ =&P(\{(\text{Barbara hit}, \text{Dianne hit}), (\text{Barbara hit}, \text{Dianne miss})\}) \\ =&p_1, \end{aligned}$$

and

$$\begin{aligned} &P(\{\text{Dianne hit}\}) \\ =&P(\{(\text{Barbara hit}, \text{Dianne hit}), (\text{Barbara miss}, \text{Dianne hit})\}) \\ =&p_2, \end{aligned}$$

Further more, assume that the event of Barbara hit and the event of Dianne hit are independent. That is

$$P(\{\text{Barbara hit}\} \cap \{\text{Dianne hit}\}) = P(\{\text{Barbara hit}\})P(\{\text{Dianne hit}\}) = p_1p_2.$$

(a)

$$\begin{aligned} &P(\{\text{both shots hit the duck}\}) \\ =&P(\{(\text{Barbara hit}, \text{Dianne hit})\}) \\ =&P(\{\text{Barbara hit}\} \cap \{\text{Dianne hit}\}) \\ =&p_1p_2. \end{aligned}$$

(b)

$$P(\{\text{Barbara's shot hit the duck}\}) = p_1.$$

Prob. 2:

(a) Prove that if $E_1, E_2, ..., E_n$ are independent events, then

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = 1 - \prod_{i=1}^{n}[1 - P(E_i)].$$

(b) Prove directly that

$$P(E|F) = P(E|FG)P(G|F) + P(E|FG^c)P(G^c|F).$$

Solution:

(a) Since $E_1, E_2, ..., E_n$ are independent events, we have

$$P(E_1^c \cap E_2^c \cap \cdots \cap E_n^c) = \prod_{i=1}^{n} P(E_i^c).$$

Then,

$$\begin{aligned}
P(E_1 \cup E_2 \cup \cdots \cup E_n) =& 1 - P((E_1 \cup E_2 \cup \cdots \cup E_n)^c) \\
=& 1 - P(E_1^c \cap E_2^c \cap \cdots \cap E_n^c) \\
=& 1 - \prod_{i=1}^{n} P(E_i^c) \\
=& 1 - \prod_{i=1}^{n}[1 - P(E_i)].
\end{aligned}$$

(b)

$$\begin{aligned}
P(E|F) =& \frac{P(EF)}{P(F)} \\
=& \frac{P((EFG) \cup (EFG^c))}{P(F)} \\
=& \frac{P(EFG)}{P(F)} + \frac{P(EFG^c)}{P(F)}, \\
& \Big(\text{since } EFG \text{ and } EFG^c \text{ are mutually exclusive,}\Big) \\
=& \frac{P(EFG)}{P(FG)}\frac{P(FG)}{P(F)} + \frac{P(EFG^c)}{P(FG^c)}\frac{P(FG^c)}{P(F)} \\
=& P(E|FG)P(G|F) + P(E|FG^c)P(G^c|F).
\end{aligned}$$

Prob. 3:

Let E, F be events with $\mathcal{P}(E) \cdot \mathcal{P}(F) > 0$.

(a) State the definition of "mutually exclusive" and "(statistically) independent".

(b) Prove or disprove with counterexample: E and F are independent if they are mutually exclusive.

(c) Prove or disprove with counterexample: E and F are mutually exclusive if they are independent.

Solution.

(a) Mutually exclusive: $\mathcal{P}(E \cap F) = 0$,
Independent: $\mathcal{P}(E \cap F) = \mathcal{P}(E) \cdot \mathcal{P}(F)$.

(b) Let S be the sample space of flipping a coin. $S = \{H, T\}$ where H means "head" and T means "tail". $\mathcal{F} = 2^S$. Let $E = \{H\}$ and $F = \{T\}$, then $E \cap F = \{\phi\}$, $\mathcal{P}(E) = \mathcal{P}(F) = \frac{1}{2}$ and $\mathcal{P}(E \cap F) = 0 \neq \frac{1}{4} = \mathcal{P}(E) \cdot \mathcal{P}(F)$. It's an example that E and F are mutually exclusive but not independent.

(c) Let S be the sample space of flipping two coins. $S = \{(H, H), (H, T), (T, H), (T, T)\}$. $\mathcal{F} = 2^S$. Let $E = \{\text{The first coin is H}\}$ and $F = \{\text{The second coin is T}\}$, then $E \cap F = \{(H, T)\}$ and $\mathcal{P}(E) \cdot \mathcal{P}(F) = \mathcal{P}(E \cap F) = \frac{1}{4} \neq 0$. It's an example that E and F are independent but not mutually exclusive.