EE306001, Probability, Fall 2012

Quiz #2, Problems and Solutions

<u>**Prob. 1:**</u> A closet contains 10 pairs of shoes, If 8 shoes are randomly selected, what is the probability that there will be

(a) no complete pair?

(b) exactly three complete pairs?

You have to specify the sample space S, the $\sigma-$algebra $\mathcal{F}$ and the probability function P. You can use the notation such as $\binom{m}{n}$ or C^m_n in your final answers.

Solution. Model all shoes in the closet as a set

$$A = \{l_i, r_i | \forall 1 \leq i \leq 10, i \in N\},$$

where l_i and r_j is a complete pair of shoes if $i = j$. The sample space S of the experiment that selects 8 shoes randomly from the closet is

$$\begin{aligned} S =& \{\omega \subset A | \ |\omega| = 8\} \\ =& \{\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\} | s_i \in A, \text{ and } s_i \neq s_j \text{ if } i \neq j, \forall 1 \leq i, j \leq 8\}, \end{aligned}$$

The σ-algebra $\mathcal{F} = 2^S$ is the power set of S. The probability function P is the counting probability measure on $\mathcal{F}$,

$$P(E) = \frac{|E|}{|S|}, \ \forall E \in \mathcal{F}.$$

(a) Let $E_1 \in \mathcal{F}$ be the event that no complete pair occurs, i.e., $E_1 = \{\text{no complete pair}\}$. Then,

$$|E_1| = \binom{10}{8} 2^8.$$

Therefore,

$$P(E_1) = \frac{|E_1|}{|S|} = \frac{\binom{10}{8} 2^8}{\binom{20}{8}}.$$

(b) Let $E_2 \in \mathcal{F}$ be the event that there are exactly three complete pairs, i.e., $E_2 = \{\text{exactly three complete pairs}\}$. Then

$$|E_2| = \binom{10}{3}\binom{7}{2} 2^2.$$

Therefore,

$$P(E_2) = \frac{|E_2|}{|S|} = \frac{\binom{10}{3}\binom{7}{2} 2^2}{\binom{20}{8}}.$$

Prob. 2: Prove Boole's inequality:

$$P(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$$

Solution. From the first inclusion-exclusion inequality, we have

$$P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i), \ \forall \ n \geq 1. \tag{1}$$

The above formula can be proved by mathematical induction as follows:

(i) **Basis step**: For $n = 1$, it is true that $P(A_1) = P(A_1)$. For $n = 2$, we have

$$\begin{aligned} P(A_1 \cup A_2) =& P(A_1) + P(A_2) - P(A_1 \cap A_2) \\ \leq& P(A_1) + P(A_2). \end{aligned}$$

(ii) **Induction step**: Assume that (**??**) is true when $n = k$, i.e.,

$$P(\bigcup_{i=1}^{k} A_i) \leq \sum_{i=1}^{k} P(A_i).$$

Then

$$P(\bigcup_{i=1}^{k+1} A_i) = P(\bigcup_{i=1}^{k} A_i \cup A_{k+1}) \leq P(\bigcup_{i=1}^{k} A_i) + P(A_{k+1}) \leq \sum_{i=1}^{k} P(A_i) + P(A_{k+1}) = \sum_{i=1}^{k+1} P(A_i).$$

By the mathematical induction theorem, we prove (**??**). Now, we define an increasing sequence of events $B_1, B_2, \ldots, B_n, \ldots$, where

$$B_n = \bigcup_{i=1}^{n} A_i, \ \forall n \geq 1.$$

Then we have

$$\lim_{n\to\infty} B_n = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n.$$

Therefore,

$$\begin{aligned} P(\bigcup_{i=1}^{\infty} A_i) =& P(\lim_{n\to\infty} B_n) = \lim_{n\to\infty} P(B_n) \ \text{ by the continuity of the probability function} \\ =& \lim_{n\to\infty} P(\bigcup_{i=1}^{n} A_i) \leq \lim_{n\to\infty} \sum_{i=1}^{n} P(A_i) \ \text{ by (\textbf{??})} \\ =& \sum_{i=1}^{\infty} P(A_i). \end{aligned}$$

Prob. 3:

Let $\{E_k\}$ be a sequence of events such that

$$\mathcal{P}(E_k) = 1,\ \forall\ k = 1, 2, \cdots$$

Show that

$$\mathcal{P}\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \mathcal{P}\Big(\bigcap_{k=1}^{\infty} E_k\Big) = 1$$

Solution.

$\mathcal{P}\Big(\bigcup_{k=1}^{\infty} E_k\Big) = 1$ is trivial since $E_1 \subset \bigcup_{k=1}^{\infty} E_k$ implies

$$1 = \mathcal{P}(E_1) \leqslant \mathcal{P}\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leqslant 1.$$

Let $F_m = \bigcap_{k=1}^{m} E_k$, then $F_1 \supset F_2 \supset F_3 \supset \cdots \supset F_m \supset \cdots$, and

$$\lim_{m\to\infty} F_m = \bigcap_{m=1}^{\infty} F_m = \bigcap_{m=1}^{\infty}\Big(\bigcap_{k=1}^{m} E_k\Big) = \bigcap_{k=1}^{\infty} E_k.$$

By the continuity property of probability function,

$$\mathcal{P}\Big(\bigcap_{k=1}^{\infty} E_k\Big) = \mathcal{P}\Big(\lim_{m\to\infty} F_m\Big) = \lim_{m\to\infty} \mathcal{P}\Big(F_m\Big) = \lim_{m\to\infty} \mathcal{P}\Big(\bigcap_{k=1}^{m} E_k\Big).$$

Since $\mathcal{P}(E_1 \cup E_2) = 1$, we have

$$\mathcal{P}(E_1 \cap E_2) = \mathcal{P}(E_1) + \mathcal{P}(E_2) - \mathcal{P}(E_1 \cup E_2) = 1 + 1 - 1 = 1.$$

Similarly, $\mathcal{P}\big((E_1 \cap E_2) \cup \mathrm{E}_3\big) = 1$, and we also have

$$\mathcal{P}(E_1 \cap E_2 \cap E_3) = \mathcal{P}(E_1 \cap E_2) + \mathcal{P}(E_3) - \mathcal{P}\big((E_1 \cap E_2) \cup \mathrm{E}_3\big) = 1 + 1 - 1 = 1.$$

Continuing this process, we have $\mathcal{P}(\bigcap_{k=1}^{n} E_k) = 1$ for all $n \geq 1$. Therefore

$$\mathcal{P}\Big(\bigcap_{k=1}^{\infty} E_k\Big) = \lim_{n\to\infty} \mathcal{P}\Big(\bigcap_{k=1}^{n} E_k\Big) = \lim_{n\to\infty} 1 = 1.$$

Another proof by using Boole's inequality.

Let $F_m = \bigcap_{k=1}^{m} E_k$

Remark. It's **incorrect** that $\mathcal{P}(E) = 1$ implies $E = S$.